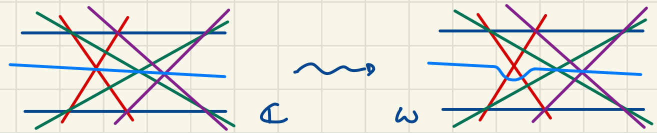

Complex curves from a symplectic perspective

Complex curves from a symplectic perspective

I will stick to the case of $\mathbb{CP}^2$ (already interesting)

Lots can be extended to the projective / Kähler setup.

Sneak peek:

$$\{\text{singular surfaces in } \mathbb{CP}^2\} \supset \{\text{symplectic curves in } \mathbb{CP}^2\} \supset \{\text{curves in } \mathbb{CP}^2\}$$

very soft — in between — very rigid

Complex curves from a symplectic perspective

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Sneak peek:

$\{$singular surfaces in $\mathbb{CP}^2\}$ $\qquad$ $\{$symplectic curves in $\mathbb{CP}^2\}$ $\qquad$ $\{$curves in $\mathbb{CP}^2\}$

very soft $\qquad$ in between $\qquad$ very rigid

Interactions go both ways:

more tools,
different viewpoint

symplectic $\qquad$ complex

fundamental examples
guiding questions

def $\omega = \frac{i}{2}\partial\bar\partial \log|z|^2$ is the Fubini-Study form

purely imaginary, since $d = \partial + \bar\partial$

What this means / what properties it has:

- $\omega \in \Omega^2(\mathbb{C}P^2)$: it measures areas of real surfaces
- $d\omega = 0$: $\int_F \omega$ does not change if we deform F.
  ($\Rightarrow [\omega] \in H^2_{dR}(\mathbb{C}P^2)$, $H^{1,1}(\mathbb{C}P^2)$)
- ω is non-degenerate : $\omega\wedge\omega > 0$ is a volume form

this is the definition of symplectic form

- ω is compatible with $i : T\mathbb{C}P^2 \circlearrowleft$: $\omega(-, i-)$ is a Riemannian metric.
  ($\Rightarrow$ ⋆ tamed by i : $\omega > 0$ on complex tangent lines
  ⋆ $\int_C \omega > 0$ $\forall$ complex curve C)

Symplectic geometry comes from classical mechanics (Hamilton, Jacobi)

Why care?

All non-singular projective and affine varieties/$\mathbb{C}$ are symplectic (Kähler)
(But there is more : cotangent bundles, 4-manifolds with non-Kähler π_1 .)

No local invariants ($\rightsquigarrow$ symplectic <u>topology</u>)

Cut-and-paste properties ($\rightsquigarrow$ contact topology)

Several descriptions : branched covers, Lefschetz fibrations, handle decompositions...

$\rightsquigarrow$ More tools to study complex varieties!

Idea: drop integrability (i.e. compatible holomorphic charts), keep ω.
↳ Gromov, Moishezon, Kharlamov, Shustin, Orevkov ...

def $J : TM \circlearrowleft$ is an **almost-complex structure** if $J^2 = -\mathrm{id}$

e.g. multiplication by i. ⚠ (rmk We are not working in the complexified TM!)

J is **compatible with** ω if $\omega(-, J-)$ is a Riemannian metric

Riemann surface

tamed by ω if $\omega > 0$ on J-invariant lines in T_pM

$u : (\Sigma, j) \to (\mathbb{CP}^2, J)$ is **J-holomorphic** if $J \circ du = du \circ j$

Cauchy-Riemann equation

rmk $C = V(F) \iff C$ is i-holomorphic.

$C \subset \mathbb{CP}^2$ is a **symplectic curve** if $C = u(\Sigma)$ where:

J ω-tame almost-complex structure, (Σ, j) a Riemann surface, u J-holomorphic.

Some things work in same way:

- Singularities of J-holomorphic curves are topologically equivalent to singularities of complex curves (i.e. same links) McDuff, Micallef-White

  ↳ ⚠ different moduli spaces!

  $\Rightarrow$ ⋆ positivity of intersections

  ⋆ existence of Milnor fibrations, resolutions. $\rightsquigarrow$ adjunction

- moduli spaces of J-holomorphic curves are well-behaved (even in families)

  ⋆ Gromov compactness

  ⋆ Oriented manifold structure (for generic $J, \{J_t\}$)

  $\rightsquigarrow$ Floer homology, Gromov-Witten invariants...

Rmk The space of ω-comp./ω-tame Js is contractible but ∞-dim'l

Some typical statements (moduli spaces)

thm (folklore?) If $F \hookrightarrow \mathbb{CP}^2$ is a smoothly embedded, oriented surface such that $\omega > 0$ on F (F is ω-symplectic), then $\exists J$ almost-complex structure on $\mathbb{CP}^2$ such that F is J-complex. I.e. F is a symplectic curve.

thm (Gromov) Let J be any compatible almost-complex structure:

- $\mathcal{M}_{\mathbb{CP}^1, J}(\mathbb{CP}^1, h) / \mathrm{Aut}(\mathbb{CP}^1) \cong \mathbb{CP}^2$ — J-holomorphic line.

Moreover, given any two points in $\mathbb{CP}^2$, $\exists!$ J-holomorphic line containing them.

⚠ $\boxed{\text{rmk This dual projective plane does } \underline{\text{not}} \text{ admit a natural almost-complex structure!}}$

- If J_1 is another almost-complex structure, L_0 is a J_0-hol^c line & L_1 a J_1-hol^c line, then L_0 is isotopic to L_1 via J_t-hol^c curves

Some things work better!

thm Suppose that $C \subset \mathbb{CP}^2$ is a symplectic curve, and let $p \in \mathrm{Sing}(C)$. Let (D,q) be a deformation of (C,p) as a germ of singularities. Then there exist $C' \subset \mathbb{CP}^2$ & a neighbourhood U of p such that

- $C \setminus U = C' \setminus U$;
- C' has a unique singularity in U, of type (D,q).

thm If S is a projective surface, M_p is the resolution/Milnor fibre of a deformation of $p \in \mathrm{Sing}(S)$, then there exist a symplectic 4-manifold X with

$$\coprod M_p \hookrightarrow X \quad \& \quad X \setminus \coprod M_p \cong S \setminus \mathrm{Sing}(S).$$

The symplectic isotopy problem:

Q Let $F_0, F_1 \hookrightarrow \mathbb{CP}^2$ be smoothly embedded, symplectic surfaces of the same degree.

Does there exist $\{F_t \overset{\omega}{\hookrightarrow} \mathbb{CP}^2\}_{t\in[0,1]}$?

Equivalently: ¿ $\exists\, u_t : (\Sigma, j_t) \longrightarrow (\mathbb{CP}^2, J_t)$ non-singular & pseudo-hol^c, & $J_1 = i$?

rmk
- If we fix $J_t = i$, then it's true: $\Delta_d \subset P^{\binom{d+2}{2}-1}$ does not disconnect.
- The degree is well-defined: $\int_F \omega / \int_L \omega$, or $\#(L\cap F)$, or $d^2 = F\cdot F$.
- The adjunction formula is pseudoholomorphic, so $\deg F_0 = \deg F_1 \Rightarrow g(F_0) = g(F_1)$.
- Answer is known (and positive) for $\deg F \leq 17$.

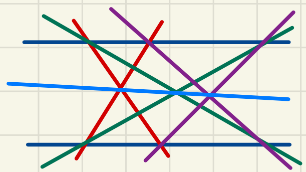
$$\begin{cases} \text{Gromov} & 1,2 \\ \text{Sikorav} & 3 \\ \text{Shevchishin} & 4-6 \\ \text{Siebert-Tian} & 7-17 \end{cases}$$

Other symplectic isotopy problems:

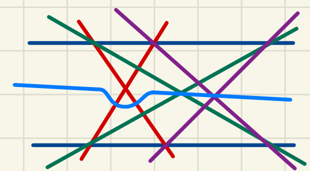
q Given $u_1 : (\Sigma, j_1) \to (\mathbb{CP}^2, J_1)$ pseudoholomorphic, does there exist
$u_t : (\Sigma, j_t) \to (\mathbb{CP}^2, J_t)$ equisingular, with $J_0 = i$?

rmk No, in many cases:
- high-genus curves with A_1-, A_2-singularities — Moishezon '94
- rational curves with reducible singularities — Orevkov ~'20
- line arrangements — folklore

ex Pappus



$\rightsquigarrow$



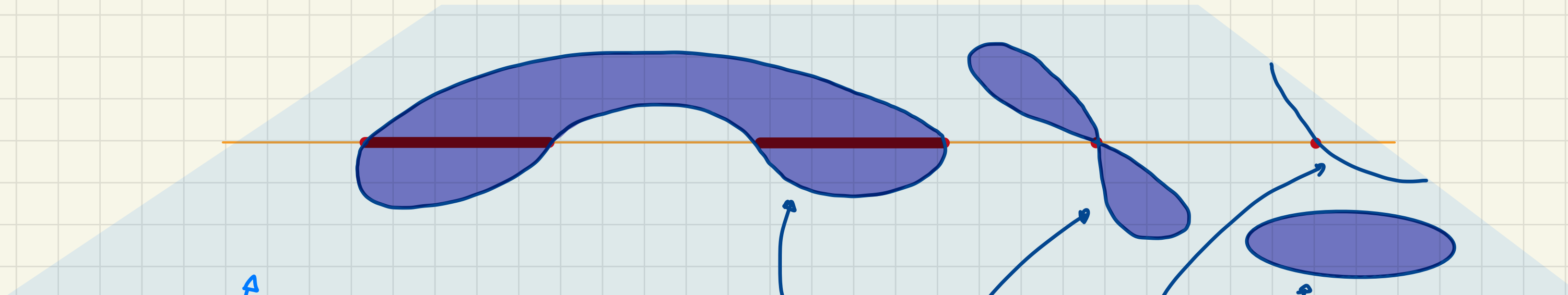
Pseudo-Pappus

thm (Ruberman, Starkston)
- Every pseudo-line arrangement in $\mathbb{RP}^2$ can be symplectified.
- There are rank-3 matroids that are not symplectically realized. (**ex** $\mathbb{F}_q P^2$)

Isotopy problems: a schematic picture

<u>rmk</u> $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2) = PGL_3(\mathbb{C})$ is connected (isomorphic curves are isotopic)

Isotopy problems : a schematic picture



degree-d

symplectic curves

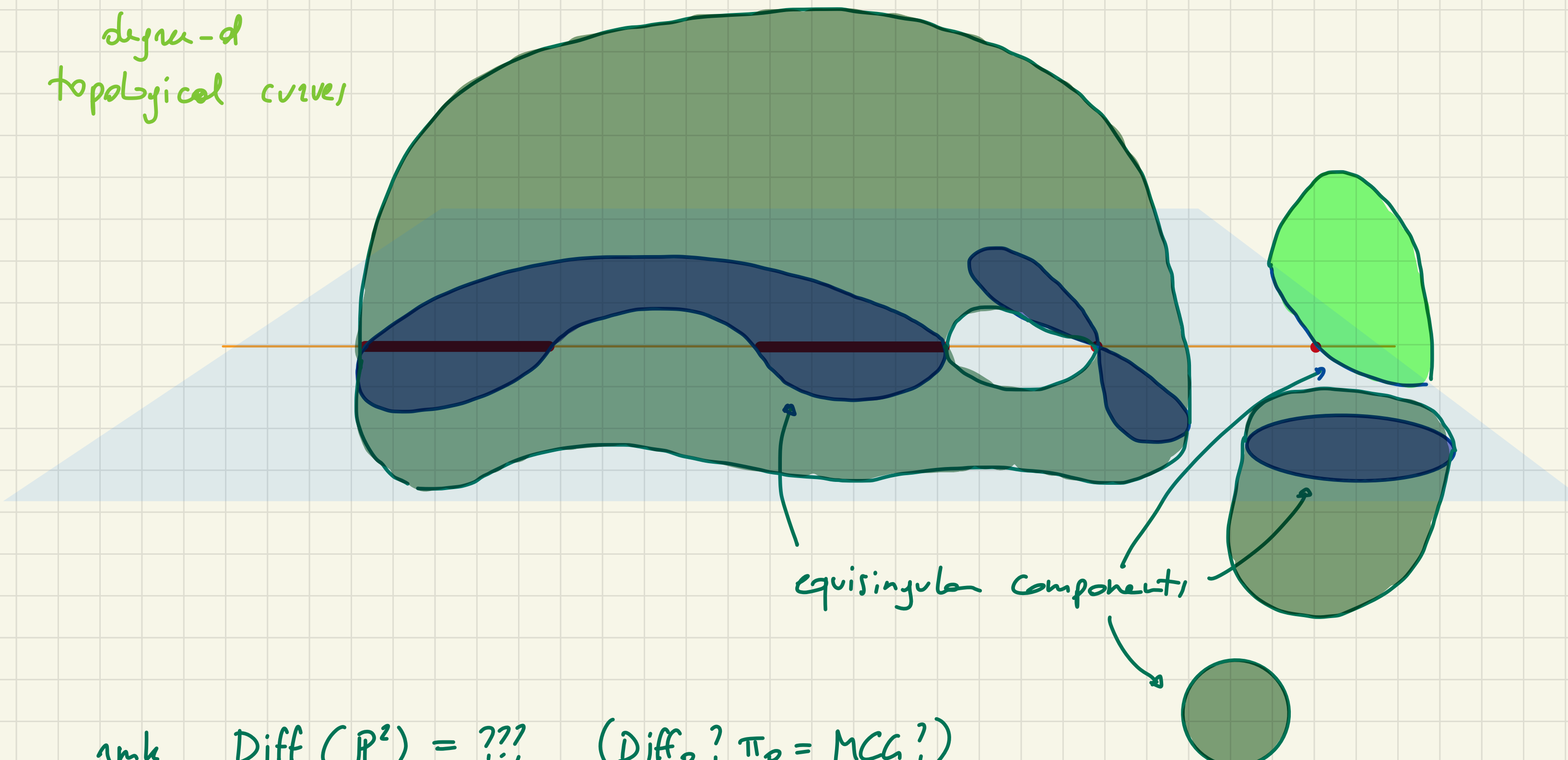
equisingular components

(Gromov)

rmk $\mathrm{Aut}_\omega(\mathbb{CP}^2) \simeq PLG_3(\mathbb{C})$ is connected (isomorphic curves are isotopic)

↳ def. retract

Isotopy problems : a schematic picture



<u>rmk</u> $\mathrm{Diff}(\mathbb{P}^2) = ???$ $(\mathrm{Diff}_0 ? \ \pi_0 = MCG ?)$

Line arrangements I

q Are there symplectic, non-complex line arrangements that do **not** come from deforming away unexpected concurrences / collinearities?

thm (Hirzebruch) If $\mathcal{A}$ is a line arrangement$/\mathbb{C}$, with t_m points of multiplicity m, $\mathcal{A}$ non-trivial ($\overset{\text{def}}{\iff}$ $\mathcal{A} \neq$  ,  $\iff t_d = t_{d-2} = 0$), then

$$t_2 + t_3 \geq d + \sum_{m \geq 4} (m-4)\, t_m \qquad (*)$$

number of points of $\mathcal{A}$ of multiplicity m.

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q Does a Hirzebruch-like inequality hold for symplectic line arrangements?

rmk • $(*)$ is proved using branched covers & BMY. Branched covers: ✓ω

q Is the BMY inequality $c_1^2 \leq 3c_2$ symplectic?

Line arrangements II

$$t_2 + t_3 \geq d + \sum_{m \geq 4} (m-4) t_m \qquad (*)$$

(*)-like inequalities hold for even/divisible/odd line arrangements.

all mult. are odd

thm (Aceto, G.) If $\mathcal{A}$ is symplectic & odd, $\mathcal{A}$ not a pencil, then

$$\sum (m-9) t_m \leq -1-7d \qquad \left(\& \sum (m-1) t_m \equiv d-1 \mod 16 \right)$$

all mult. & the degree are divisible by p

If $\mathcal{A}$ is symplectic & p-divisible, $\mathcal{A}$ not a pencil, then

$$\sum \Big(m - \frac{6p}{p+1}\Big) t_m \leq d$$

$= 4$ if $p=2$.

<u>Cor</u>

$$\text{minimal multiplicity} \leq \begin{cases} 7 & \text{if } \mathcal{A} \text{ is odd} \\ 4 & \text{if } \mathcal{A} \text{ is 2-divisible} \\ 6 & \text{if } \mathcal{A} \text{ is } p\text{-divisible} \end{cases} \qquad (!!)$$

Rational cuspidal curves (RCCs)

$p_g = 0$, irreducible singularities

homeomorphic to $\mathbb{CP}^1 \cong S^2$

Rational cuspidal curves (RCCs)

q Do there exist: - symplectic RCCs that are not equisingular to a complex one?

- symplectic RCCs with more than four cusps?
- Zariski pairs of (symplectic) RCCs?

↳ pairs of curves that are equisingular, but not equisingularly isotopic

rmk • The classification of complex RCCs is known only subject to Palka's negativity conjecture (Palka, Pełka).

• The answer to the questions above is known to be negative if:

deg ≤ 7 (G., Starkston; G., Kütle)

one singularity, one Puiseux pair (G., Starkston; Fernández de Bobadilla, Luengo, Melle Hernández, Némethi)

Invariants: the complement

proper, complex surfaces in $\mathbb{C}^N$

Complex curve complements are Stein surfaces, by the Veronese embedding.

rmk Eliashberg gave a topological characterisation of Stein surfaces / Stein homotopy

q Are complements of symplectic curves Stein surfaces?

i.e. π_1s of quasi-projective varieties

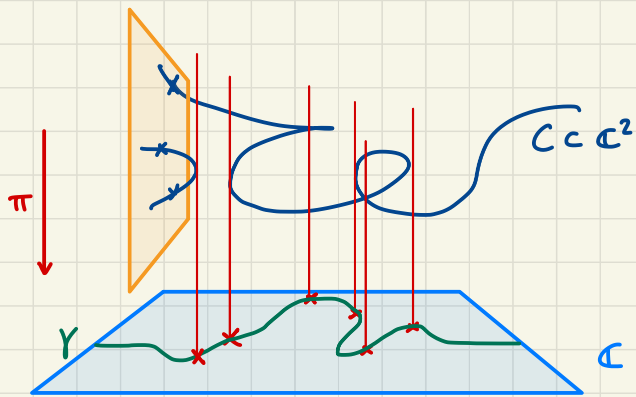
q Are fundamental groups of complements of symplectic curves quasi-projective?

rmk Alexander polynomials (Libgober) do not seem to give easy obstructions (Arnaude, G.)

.. but maybe characteristic varieties can? (Arapura)

Braid monodromy I

Zariski, van Kampen, Chisini, Moishezon, Teicher . .



Given $C \subset \mathbb{CP}^2$, choose $p \in \mathbb{CP}^2 \setminus C$,

$\pi : \mathbb{CP}^2 \setminus \{p\} \longrightarrow \mathbb{CP}^1$

Remove a point in $\mathbb{CP}^1$, so we get $\mathbb{C}$

$S := \pi(\mathrm{Sing}(\pi|_C))$ is a finite set in $\mathbb{C}$.

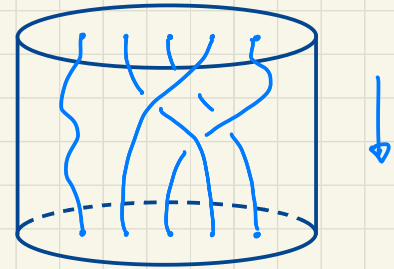
Choose a path γ_0, that agrees with $\mathbb{R} \subset \mathbb{C}$ when $|\mathrm{Re}\, \gamma_0| \gg 0$, and that passes through each point in S.

For almost all $z \in \gamma_0$, $\pi^{-1}(z) =: \mathbb{C}_z$ intersects C transversely in $d = \deg C$ points.

Following these points around along γ_0 gives a sequence of d-braids in B_d.

Braid monodromy II

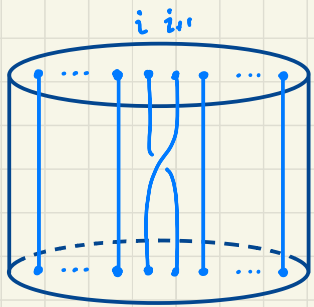
def A **d-braid** is an element of $B_d = \pi_1(\mathrm{Sym}^d(D^2) \setminus \Delta, \underline{x})$:



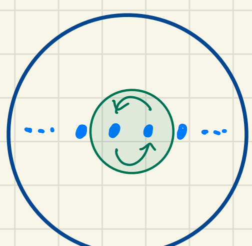
$\sigma_3\sigma_2\sigma_4\sigma_3^{-1} \in B_5$

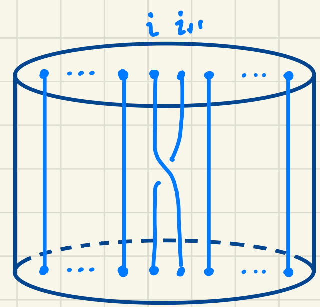
Prop $B_d \cong \mathrm{Artin}(A_{d-1}) \cong \mathrm{Diff}^+(D^2, d \text{ points}) / \mathrm{Diff}_0(D^2, d \text{ points})$

$$\cong \left\langle \sigma_1, \dots, \sigma_{d-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j|>1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \end{array} \right\rangle$$

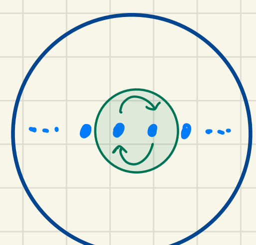


σ_i





σ_i^{-1}



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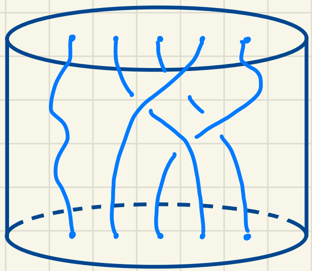

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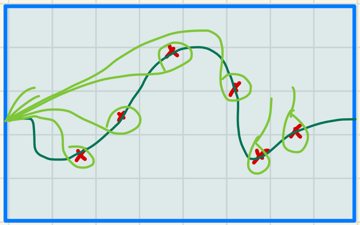
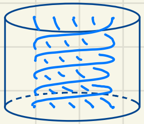
$$\cong \left\langle \sigma_1, \dots, \sigma_{d-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & |i-j|>1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \end{array} \right\rangle$$



Looking at $(\pi^{-1}(\gamma_i), \pi^{-1}(\gamma_i) \cap C)$ gives a sequence of d-braids $(\beta_1, \dots, \beta_s)$, with the following two properties:

- each β_i is the conjugate of a local monodromy
  (e.g. $\beta_i \sim \sigma_1$ for vertical tangencies of C, σ_1^2 for double points..)
- If the line at infinity $L_\infty \pitchfork C$, then $\prod \beta_i = \Delta_d^2$

↳ full twist on d strands:



Braid monodromy III

« The braid monodromy of a curve determines its embedded topology. »

thm (Artal Bartolo, Carmona Ruber, Cogolludo Agustín) (C, L_∞, p) & (C', L'_∞, p') have the same braid monodromy iff $\exists\, F : \mathbb{CP}^2 \circlearrowleft^{\text{homeo}^+}$ s.t. $F(C, L_\infty, p) = F(C', L'_\infty, p')$

thm (Kulikov, Kharlamov) $\{\text{Symplectic curves}\}/\text{equisingular symplectic isotopy}$

$$\updownarrow 1:1$$

$\{\text{Factorizations of } \Delta_d^2 \text{ into conjugates of algebraic monodromies}\}/\text{Hurwitz moves}$

H1: global conjugation

H2: $\beta_i, \beta_{i+1} \longleftrightarrow \beta_{i+1}^{\beta_i}, \beta_i$

rmk Every non-singular symplectic curve is isotopic to a complex one iff $\exists!$ factorization $\Delta_d^2 = \prod_{i=1}^{d(d-1)} \sigma_1^{\gamma_i}$ up to Hurwitz moves.

Birational transformations

Blow-ups & blow-downs also work in the symplectic category.

rmk Technical point: there is a parameter involved in the construction: $\int_E \omega$.

This (& the origin of this parameter) complicate the matters slightly...

One can define birational transformations of symplectic 4-manifolds

& of their symplectic curves.

ex Orevkov's octic is birational to a pseudo-Desargues configuration.

rmk One can also define the Kodaira dimension of a symplectic 4-manifold (M, ω)

in terms of $K_\omega \cdot [\omega]$ and $K_\omega \cdot K_\omega$ (T.-J. Li), and get classification

results for $\kappa = -\infty$ (rational or ruled). $\big(\underbrace{\text{Taubes, Li}}_{\text{Enriques-Kodaira}}, \underbrace{\text{McDuff, Gromov..}}_{\text{Castelnuovo}}\big)$.

Branched covers

thm (folklore?) If (M,ω) is a symplectic manifold, $B \subset M$ is a symplectic surface, $p : \tilde{M} \to M$ is a cover that ramifies over B, then $\tilde{M}$ admits a symplectic structure $\tilde{\omega}$ compatible with ω, p, B:

- $p^* \omega = \tilde{\omega}$ away from B and $p^{-1}(B)$
- $p^{-1}(B)$ is $\tilde{\omega}$-symplectic.

thm (Auroux) Every symplectic 4-manifold is a branched cover of $\mathbb{CP}^2$, ramified over a singular symplectic surface with singularities of type

A_1, $\bar{A}_1$, A_2 — not J-holomorphic: $\{z\bar{w} = 0\}$!

q Are points of type $\bar{A}_1$ necessary?