AGATHA - Algebraic and Geometric Aspects of the Theory of Hyperplane Arrangements 2026
(Krakow)

What does the real structure of a hyperplane arrangement tell us.

2026/06/02
Masahiko Yoshinaga (Osaka)

Plan

1. Real str. and topology

Zaslavsky, Deligne, Salvetti, Oriented matroids, Aomoto ...

2. Varchenko-Gelfand algebra

Varchenko-Gelfand, Orlik-Solomon, ...

3. Main results and Problems

(j.w. Yukino Yagi, arXiv:2509.19905, to appear in IMRN)

1. Real str. and topology

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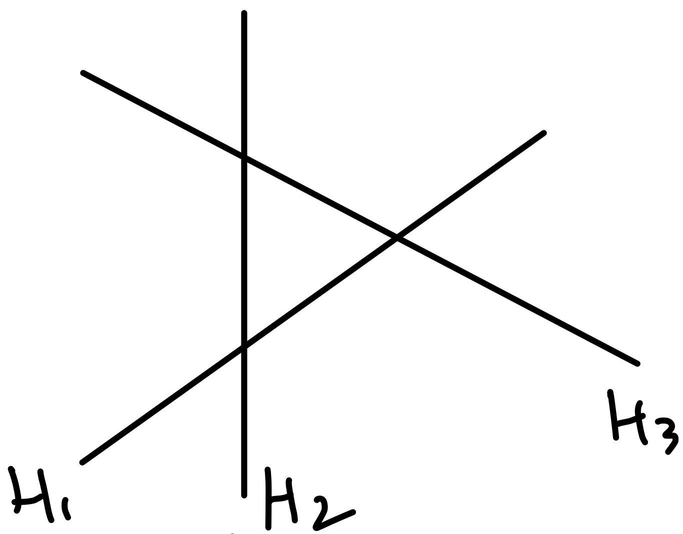
<u>Setting</u> $\mathcal{A} = \{H_1, \cdots, H_n\}$: arrangement in $\mathbb{R}^\ell = V$

i.e. $H_i \subseteq \mathbb{R}^\ell$: affine hyperplane.

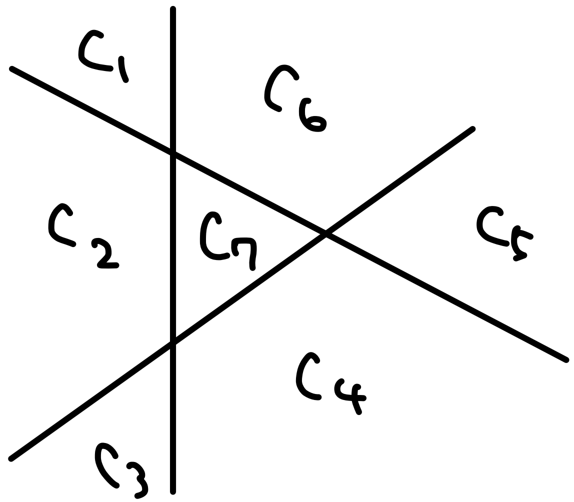
<u>Notation</u> $M = M(\mathcal{A}) := \mathbb{C}^\ell \setminus \bigcup_{i=1}^{n} H_i \otimes \mathbb{C}$: complexified complement.

$ch(\mathcal{A}) := \{ C :$ conn. comp (chamber) of $\mathbb{R}^\ell \setminus \bigcup_{i=1}^{n} H_i \}$.

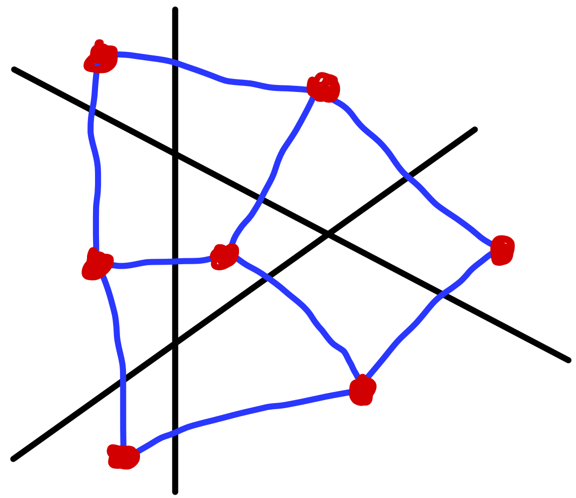
$T(\mathcal{A}) := (ch(\mathcal{A}), \{(C, C') :$ C and C' are adjacent $\})$: adjacency graph.



$\mathcal{A} = \{H_1, H_2, H_3\}$

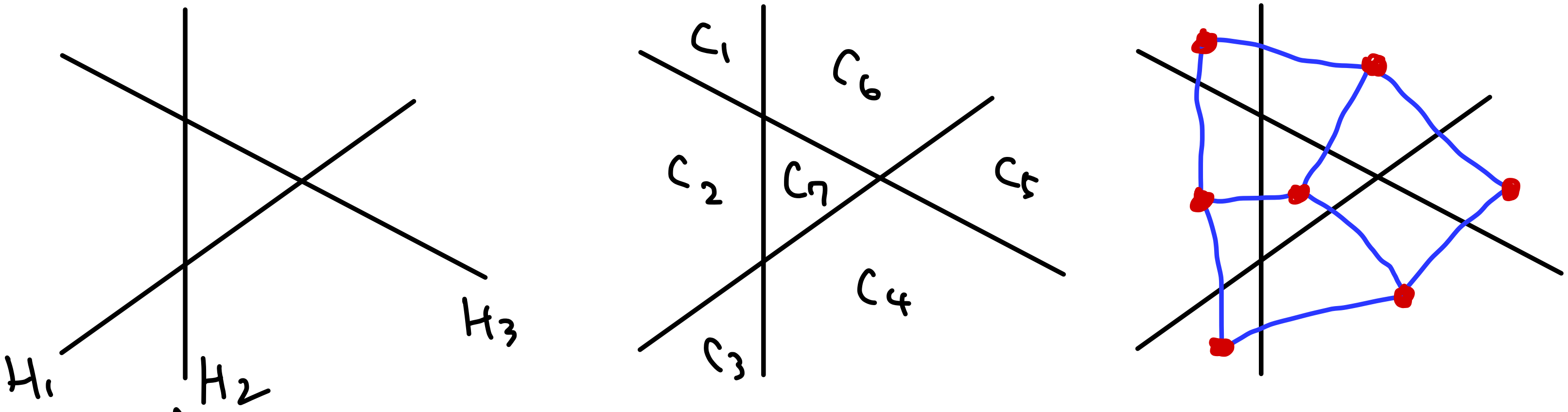


$ch(\mathcal{A}) = \{C_1, C_2, \cdots, C_7\}$



$T(\mathcal{A})$

1. Real str. and topology



$\mathcal{A} = \{H_1, H_2, H_3\}$ $\quad$ $ch(\mathcal{A}) = \{C_1, C_2, \cdots, C_7\}$ $\quad$ $T(\mathcal{A})$

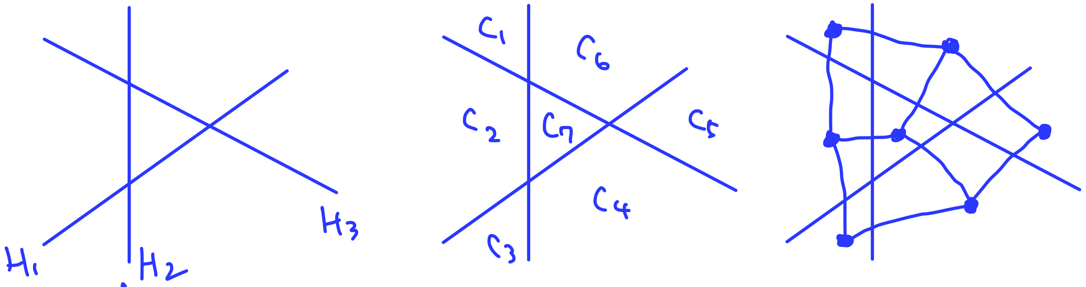
Real structure is related to the topology of $M = \mathbb{C}^\ell \setminus \bigcup H_i \otimes \mathbb{C}$.

<u>Fact</u> (Zaslavsky 1975)

$$\# ch(\mathcal{A}) = \sum_{i \geq 0} b_i(M)$$

$$\# \underbrace{\text{bdd ch}}_{\text{bounded chambers}}(\mathcal{A}) = (-1)^\ell \cdot \sum_{i \geq 0} (-1)^i \cdot b_i(M).$$

1. Real str. and topology



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Real structure is related to the topology of $M = \mathbb{C}^\ell \setminus \bigcup H_i \otimes \mathbb{C}$.

Def $\mathcal{A} = \{H_1, \cdots, H_n\}$ is central $:\overset{\text{def}}{\Longleftrightarrow} 0 \in \bigcap_{i=1}^{n} H_i$.

Def A central arr $\mathcal{A}$ is simplicial $:\overset{\text{def}}{\Longleftrightarrow}$ Every chamber is a simplicial cone.

(see the title page of booklet.) $\quad$ # of walls $= \ell$.

Fact (Deligne 1972) If $\mathcal{A}$ is simplicial, then $M = M(\mathcal{A})$ is $K(\pi, 1)$.

$:\Longleftrightarrow \pi_k(M) = 0$ for $k \geq 2$

$\Longleftrightarrow$ the univ. cover $\tilde{M}$ is contractible.

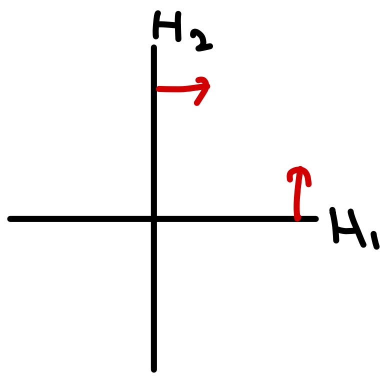
Rem. (possibly infinite, locally finite) Affine simplicial arr. are conjectured to be $K(\pi, 1)$. (cf. Cuntz's talk.)

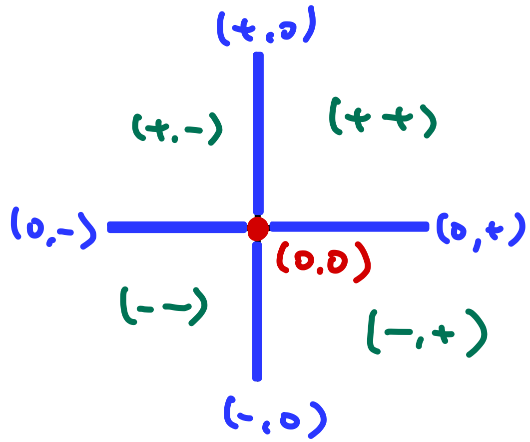
Fact (Paolini - Salvetti 2021) Affine Weyl arr.'s are $K(\pi, 1)$.

1. Real str. and topology

Let $\mathcal{A} = \{H_1, \cdots, H_n\}$ be an arr. in $\mathbb{R}^\ell$.

Fix positive (resp. negative) half-space H_i^+ (resp. H_i^-) $\subset \mathbb{R}^\ell$.





<u>Def</u> $\mathcal{F}(\mathcal{A})$ is the poset associated to the stratification defined by $\mathcal{A}$.

Below, $\mathcal{A}$: central.

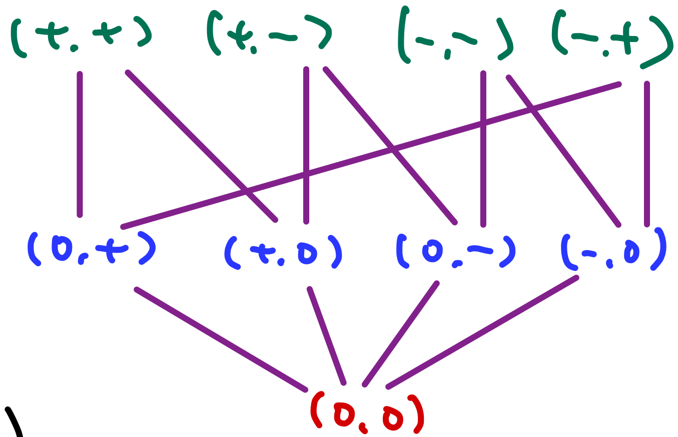
<u>Fact</u> The following notions are equiv.

(i) Oriented matroid of $\mathcal{A}$. (up to re-ori.)

(ii) The adjacency graph $T(\mathcal{A})$.

(iii) The face poset $\mathcal{F}(\mathcal{A})$.

$\mathcal{F}(\mathcal{A})$:



<u>Fact.</u> (Salvetti 1987)

$\mathcal{F}(\mathcal{A})$ determines the homotopy type of $M = M(\mathcal{A})$.

<u>Rem</u> Later, Bjorner-Ziegler proved that $\mathcal{F}(\mathcal{A})$ recovers the homeo-type of M.

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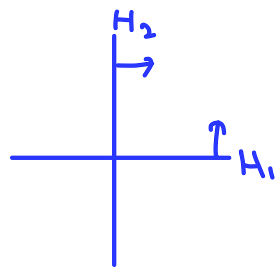

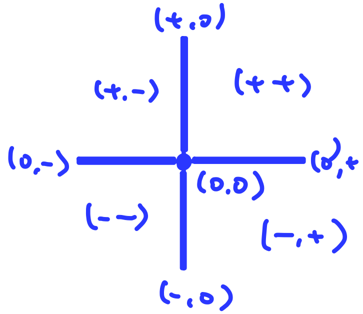

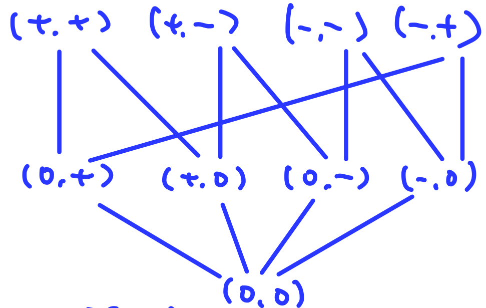

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In short, the oriented matroid (or $T(\mathcal{A})$, or $\mathcal{F}(G)$) determines the topology of $M = M(\mathcal{A})$.

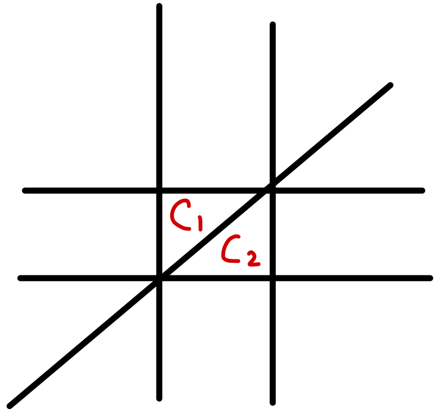
"Real str. knows everything".

<u>Fact</u> (Aomoto, Kohno, 1986)

Let $\mathcal{L}$ be a $\text{rank}_{\mathbb{C}}$ 1 local system on $M = M(\mathcal{A})$.

Then

$$H^k(M, \mathcal{L}) = \begin{cases} 0 & k \neq \ell \\ \bigoplus_{C \in \mathrm{bddch}(\mathcal{A})} \mathbb{C} \cdot [C], & k = \ell. \end{cases}$$



$$H^2(M, \mathcal{L}) = \mathbb{C} \cdot [C_1] \oplus \mathbb{C} \cdot [C_2].$$

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"Real str. knows everything".

<u>Fact</u> (Y. 2026) $\mathcal{A} = \{H_1, \cdots, H_n\}$: central arr. in $\mathbb{R}^\ell$.

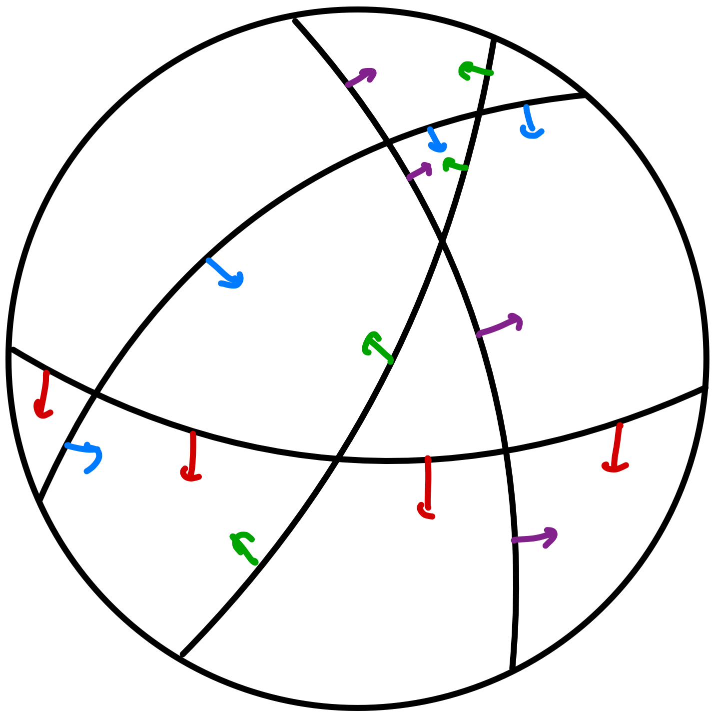
Suppose that $\exists$ half-spaces $H_1^{\varepsilon_1}, H_2^{\varepsilon_2}, \cdots, H_n^{\varepsilon_n}$, $(\varepsilon_1, \cdots, \varepsilon_n \in \{+, -\})$

(i) $\bigcap_{i=1}^{n} H_i^{\varepsilon_i} = \phi$,

(ii) For each $x \in \mathbb{R}^\ell \setminus \{0\}$,

$$\bigcap_{H_i \ni x} H_i^{\varepsilon_i} \neq \phi$$

Then $\pi_\ell(M) \neq 0$.



<u>Rem.</u> T. Saito and S. Yamagata recently applied this result to prove that Manin-Schechtmann's arr. is not $K(\pi, 1)$ (arXiv:2605.14536)

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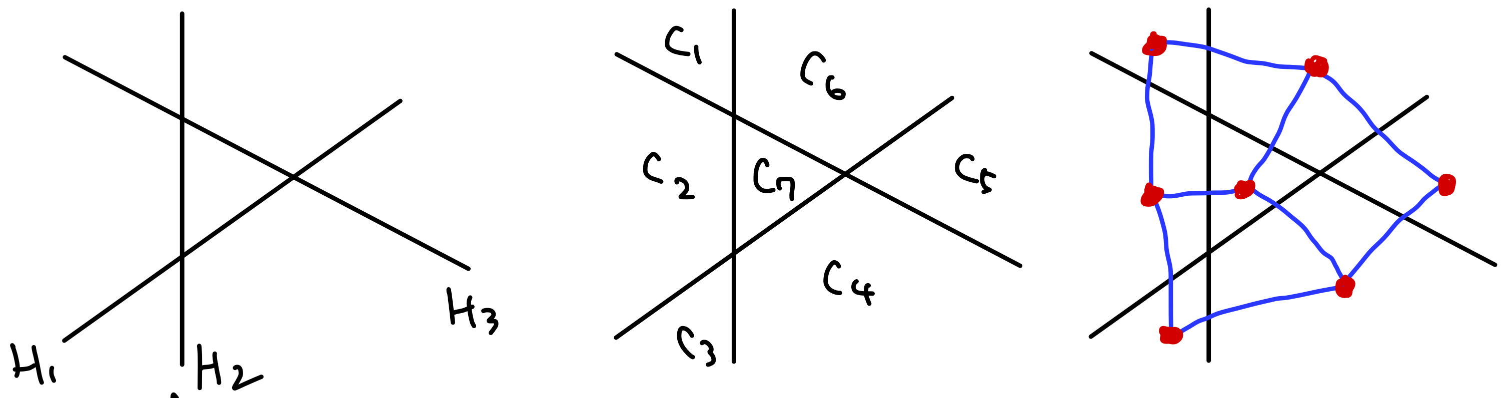

$\mathcal{A}=\{H_1, H_2, H_3\}$ $\quad$ $ch(\mathcal{A})=\{C_1, C_2, \cdots, C_7\}$ $\quad$ $\mathcal{T}(\mathcal{A})$

In this (and next) section, $\mathcal{A}$ is central.

Def Let R be an integral domain.

$VG(\mathcal{A}) := R^{ch(\mathcal{A})} = \{f : ch(\mathcal{A}) \to R\}$: the ring of functions on $ch(\mathcal{A})$.

Rem As R-algebras, $VG(\mathcal{A}) \cong R^{\# ch(\mathcal{A})}$.

2. Varchenko - Gelfand algebra

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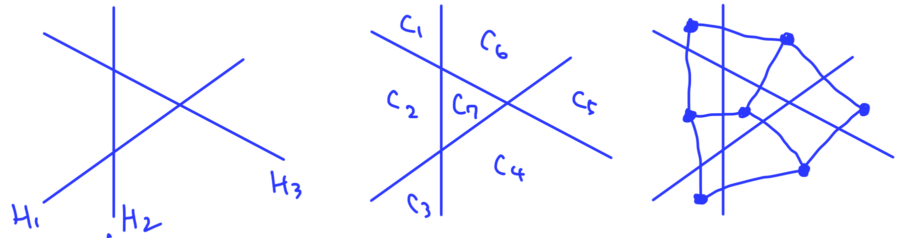
$\Gamma(\mathcal{A}) := (ch(\mathcal{A}), \{(C, C') : C \text{ and } C' \text{ are adjacent}\})$: adjacency graph.



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<u>Rem</u> As R-algebras, $VG(\mathcal{A}) \cong R^{\# ch(\mathcal{A})}$.

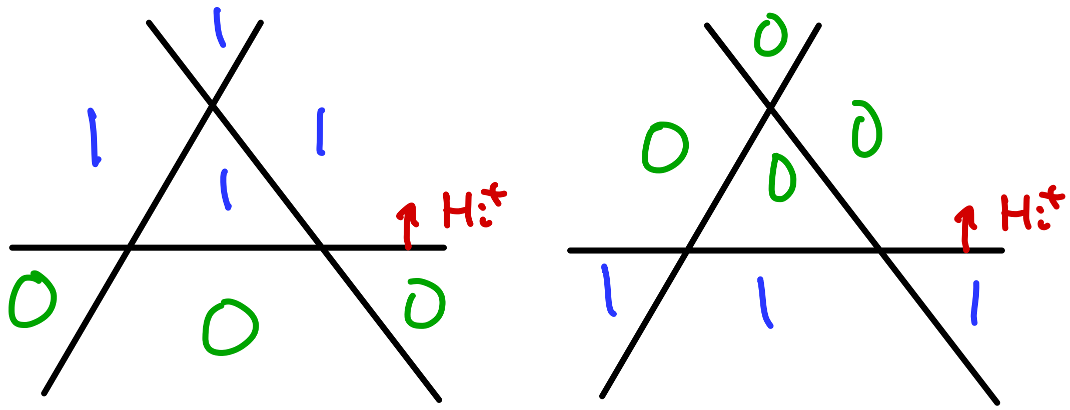
For each $C \in ch(\mathcal{A})$, let $1_C \in VG(\mathcal{A})$ be the characteristic function of C.

<u>Def</u> (Heaviside function)

$$x_i^+ := \sum_{C \subseteq H_i^+} 1_C ,$$

$$x_i^- := \sum_{C \subseteq H_i^-} 1_C ,$$

are called the Heaviside function.



x_i^+ $\quad x_i^-$

<u>Rem.</u> $VG(\mathcal{A})$ is generated by $x_i^\pm$.

2. Varchenko-Gelfand algebra

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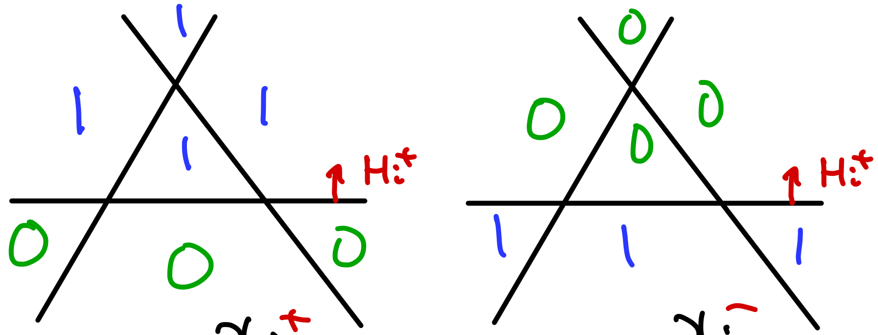
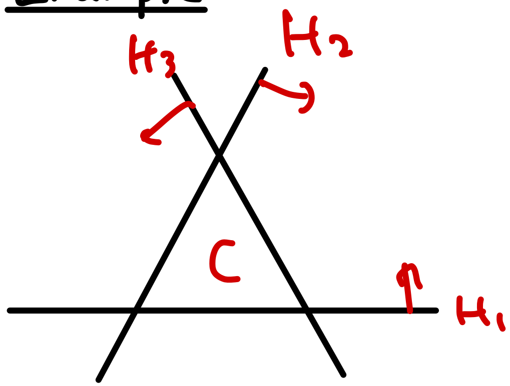
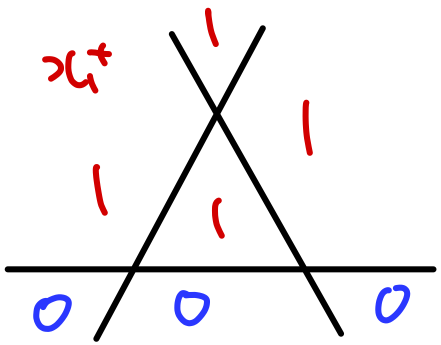
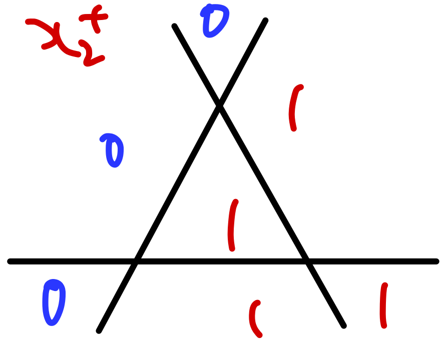


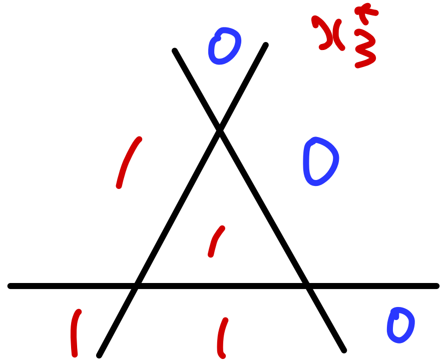
x_i^+ $\qquad$ x_i^-

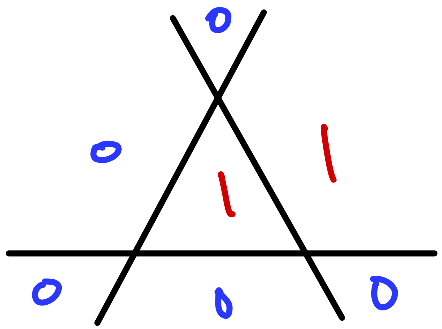
Example



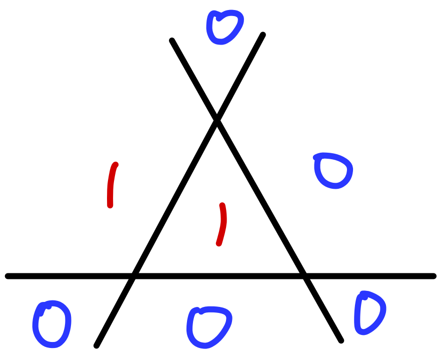




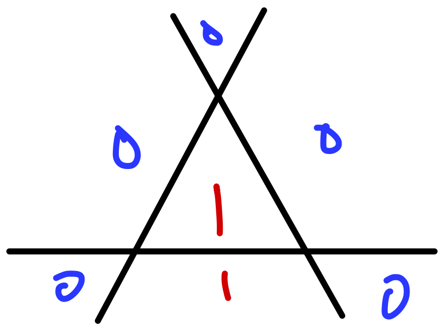




$x_1^+ x_2^+$



$x_1^+ x_3^+$



$x_2^+ x_3^+$

$$x_1^+x_2^+ + x_1^+x_3^+ + x_2^+x_3^+ - x_1^+ - x_2^+ - x_3^+ + 1$$

$$= 1_C$$

(deg=2 in Heavisides)

2. Varchenko - Gelfand algebra

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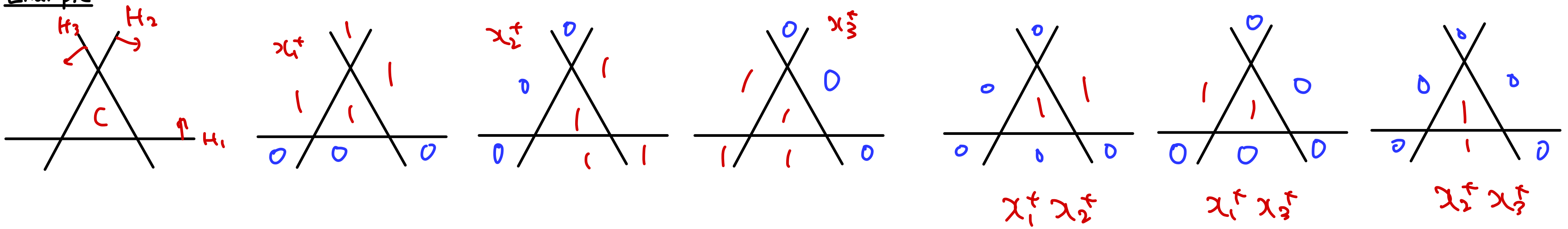

$$x_1^+x_2^+ + x_1^+x_3^+ + x_2^+x_3^+ - x_1^+ - x_2^+ - x_3^+ + 1 = 1_C$$ (deg=2 in Heavisides)

Def (VG-filtration)

$F_k VG(\mathcal{A}) := \{ f \in VG(\mathcal{A}) \mid f$ is expressed by $x_1^+, \cdots, x_n^+$ with $\deg \leq k \}$ $(k \geq 0)$

$0 =: F_{-1} \subset F_0 = R \subset F_1 \subset F_2 \subset \cdots$ gives a filtered algebra $(F_i \cdot F_j \subset F_{i+j})$.

Def (graded VG algebra)

$VG^k(\mathcal{A}) := F_k VG(\mathcal{A}) / F_{k-1} VG(\mathcal{A})$. $\quad VG^\bullet(\mathcal{A}) := \bigoplus_{k \geq \ell} VG^k(\mathcal{A})$.

Fact (Varchenko - Gelfand 1987) $\mathcal{A}$: arr. in $\mathbb{R}^\ell$.

(1) $F_\ell = F_{\ell+1} = \cdots = VG(\mathcal{A})$ (every $f \in VG(\mathcal{A})$ is $\deg \leq \ell$ w.r.t. Heavisides)

(2) rank $VG^k(\mathcal{A}) = b_k(M(\mathcal{A}))$ ($VG^k(\mathcal{A}) \cong H^k(M, R)$ as R-modules)

(3) When $R = \mathbb{F}_2$, $VG^\bullet(\mathcal{A}) \cong H^\bullet(M, \mathbb{F}_2)$ as graded $\mathbb{F}_2$-algebras.

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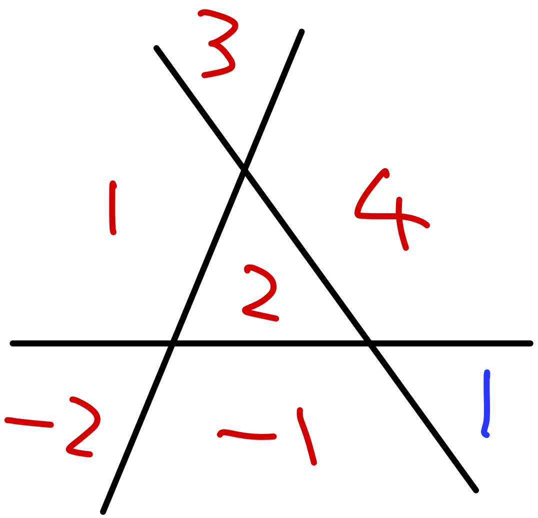
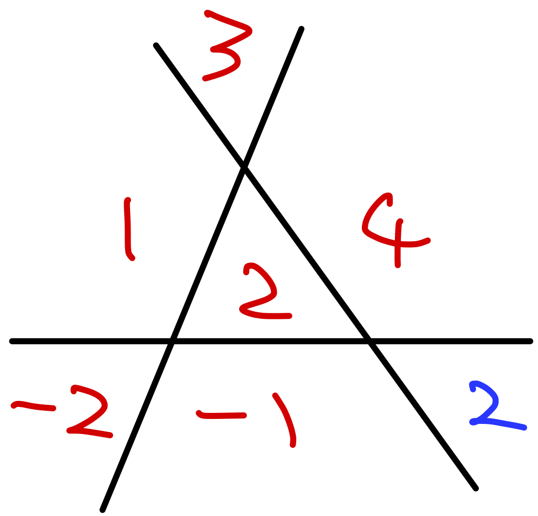
Quiz Which is in $F_1 \mathcal{VG}(\mathcal{A})$?





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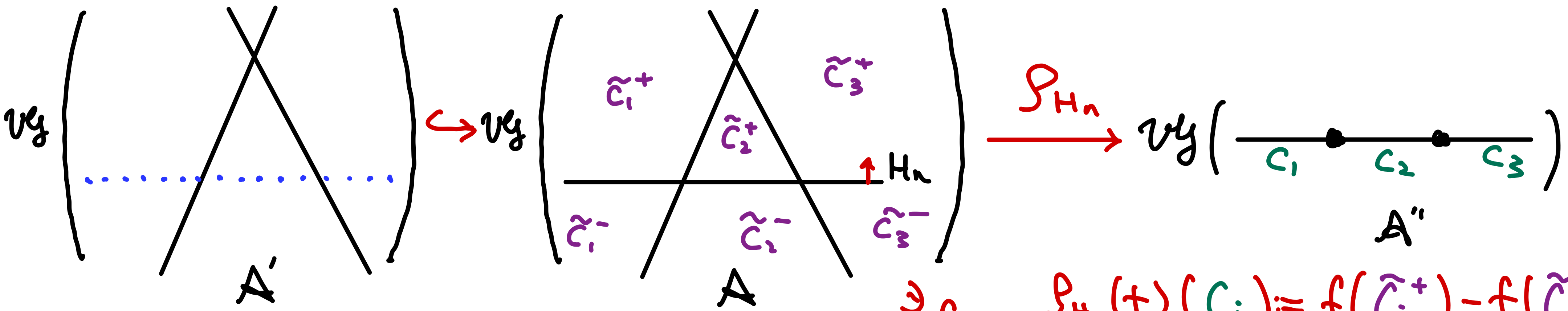
(3) When $R = \mathbb{F}_2$, $VG^\bullet(\mathcal{A}) \cong H^\bullet(M, \mathbb{F}_2)$ as graded $\mathbb{F}_2$-algebras.

Important tool: deletion-restriction exact seq. among

$\mathcal{A} = \{H_1, \cdots, H_n\}$, $\mathcal{A}' = \mathcal{A} \setminus \{H_n\}$, $\mathcal{A}'' = \mathcal{A}^{H_n}$ (induced arr. on H_n)

<u>Fact</u> $\exists$ exact seq. of R-modules: $0 \to VG(\mathcal{A}') \to VG(\mathcal{A}) \xrightarrow{\rho_{H_n}} VG(\mathcal{A}'') \to 0$



$\ni f \quad \rho_{H_n}(f)(C_i) := f(\tilde{C}_i^+) - f(\tilde{C}_i^-)$

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<u>Fact</u> $\mathcal{A}$: arr. in $\mathbb{R}^\ell$. (1) $F_\ell = \mathcal{VG}(\mathcal{A})$ (2) $\operatorname{rank} VG^k(\mathcal{A}) = b_k(M(\mathcal{A}))$ (3) When $R = \mathbb{F}_2$, $VG^\bullet(\mathcal{A}) \cong H^\bullet(M, \mathbb{F}_2)$ as graded $\mathbb{F}_2$-algebras.

Important tool: deletion-restriction exact seq.

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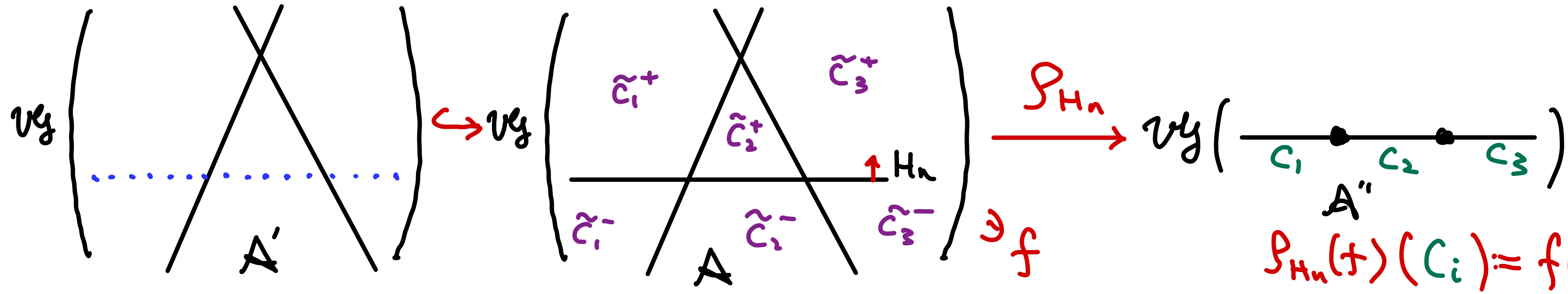


$\rho_{H_n}(f)(C_i) := f(\tilde{C}_i^+) - f(\tilde{C}_i^-)$

This exact seq. is compatible with VG-filtration:

<u>Fact</u> The above exact seq. induces the following exact sequences.

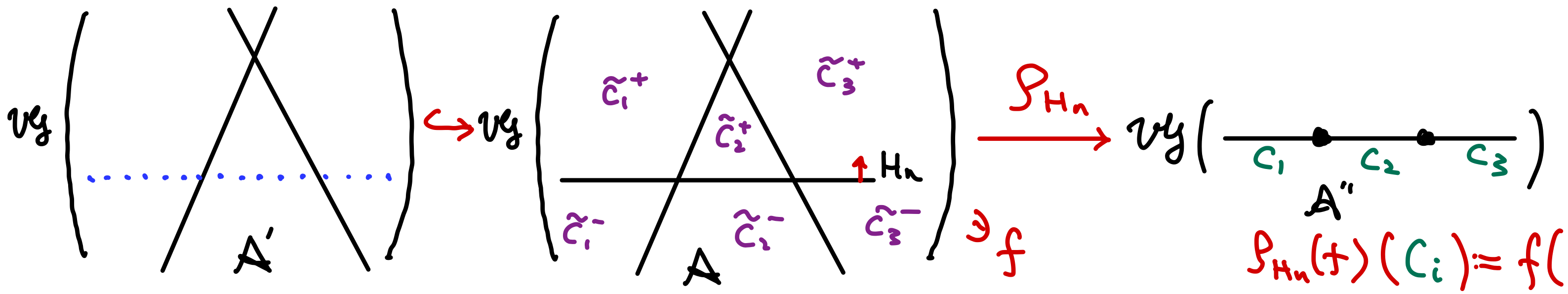
(1) $0 \to F_k(\mathcal{A}') \to F_k(\mathcal{A}) \xrightarrow{\rho_{H_n}} F_{k-1}(\mathcal{A}'') \to 0$

(2) $0 \to VG^k(\mathcal{A}') \to VG^k(\mathcal{A}) \to VG^{k-1}(\mathcal{A}'') \to 0$

cf $0 \to H^k(M(\mathcal{A}')) \to H^k(M(\mathcal{A})) \xrightarrow{\mathrm{Res}} H^{k-1}(M(\mathcal{A}'')) \to 0$

2. Varchenko-Gelfand algebra

<u>Fact</u> $\exists$ exact seq. of R-modules : $0 \to VG(\mathcal{A}') \to VG(\mathcal{A}) \xrightarrow{\rho_{H_n}} VG(\mathcal{A}'') \to 0$



$\exists f$ $\quad \rho_{H_n}(f)(C_i) := f(\tilde{C}_i^+) - f(\tilde{C}_i^-)$

<u>Fact</u> The above exact seq. induces the following exact sequences.

(1) $0 \to F_k(\mathcal{A}') \to F_k(\mathcal{A}) \xrightarrow{\rho_{H_n}} F_{k-1}(\mathcal{A}'') \to 0$

(2) $0 \to VG^k(\mathcal{A}') \to VG^k(\mathcal{A}) \to VG^{k-1}(\mathcal{A}'') \to 0$

Furthermore, we also have the following strong compatibility.

<u>Fact</u> Let $f \in VG(\mathcal{A})$. Then $f \in F_k VG(\mathcal{A}) \iff \rho_{H_i}(f) \in F_{k-1}(\mathcal{A}^{H_i})$ for $\forall i = 1, \cdots, n$.

<u>Cor.</u> Let $f \in VG(\mathcal{A})$. Then $f \in F_1 VG(\mathcal{A}) \iff \rho_{H_i}(f)$ is constant for $\forall i = 1, \cdots, n$.

Now we can answer to the Quiz.

2. Varchenko-Gelfand algebra

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<u>Fact</u> Let $f \in \mathcal{VG}(\mathcal{A})$. Then $f \in F_k \mathcal{VG}(\mathcal{A}) \iff \rho_{H_i}(f) \in F_{k-1}(\mathcal{A}^{H_i})$ for $\forall i = 1, \cdots, n$.

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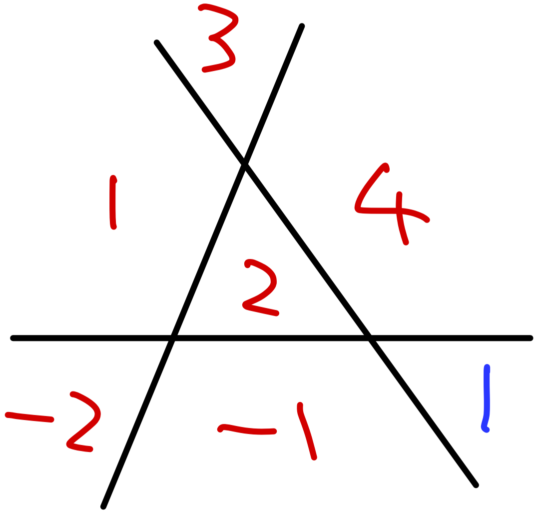
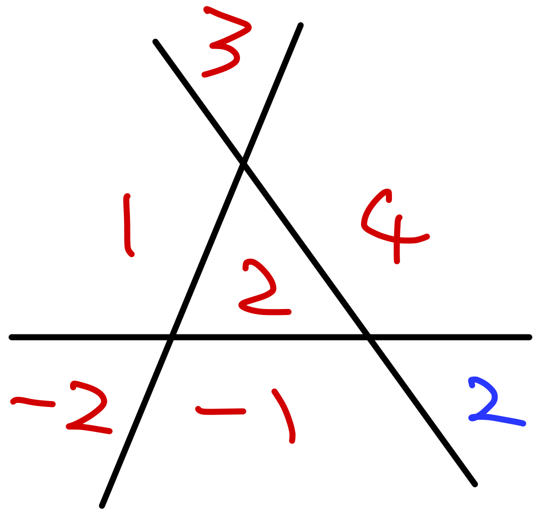
<u>Quiz</u> Which is in $F_1 \mathcal{VG}(\mathcal{A})$?





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Cor. Let $f \in \mathcal{VG}(\mathcal{A})$. Then $f \in F_1\mathcal{VG}(\mathcal{A}) \iff \rho_{H_i}(f)$ is constant for $\forall i = 1, \cdots, n$.

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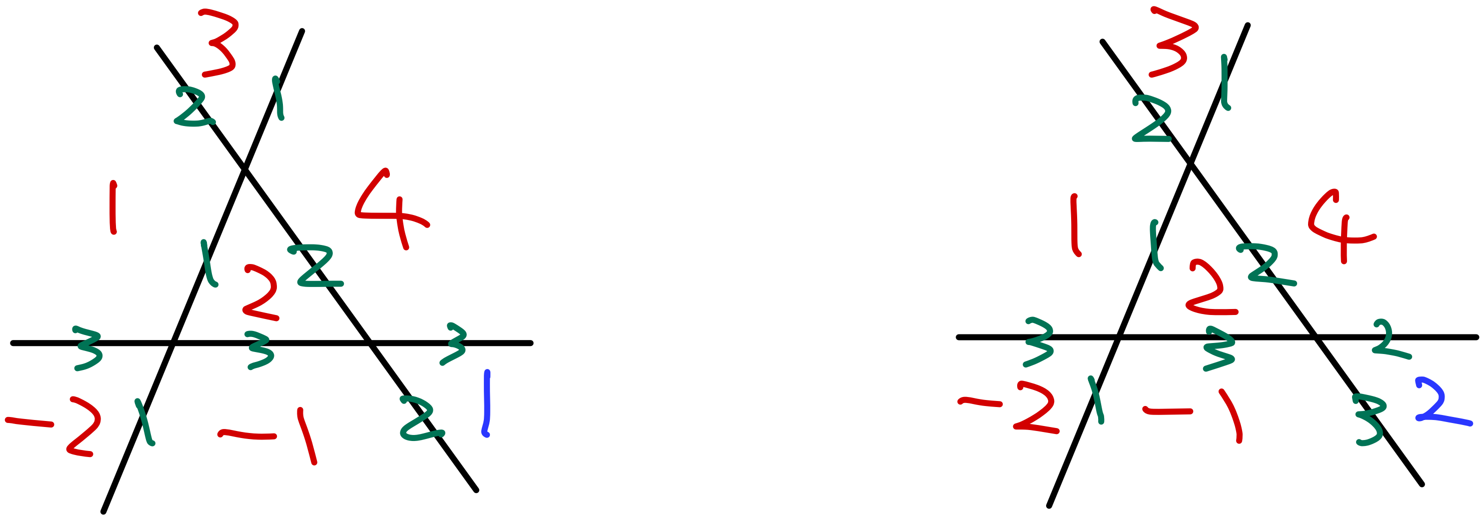

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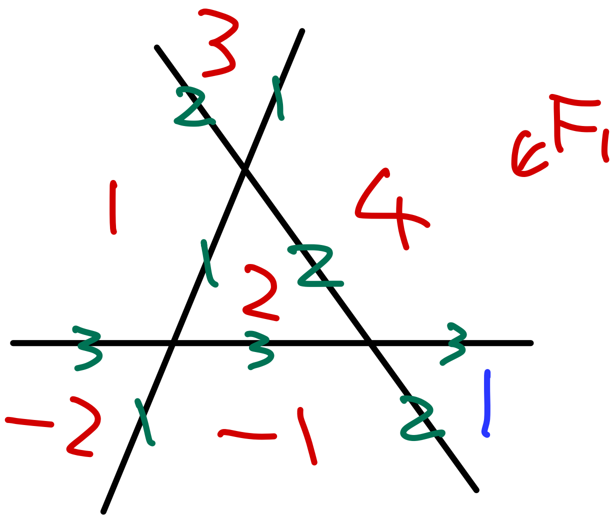
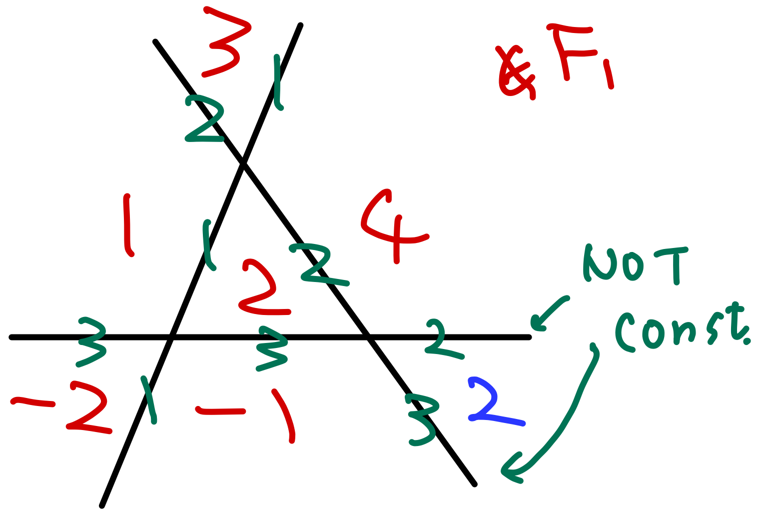




3. Main results and Problems.

3. Main results and Problems.

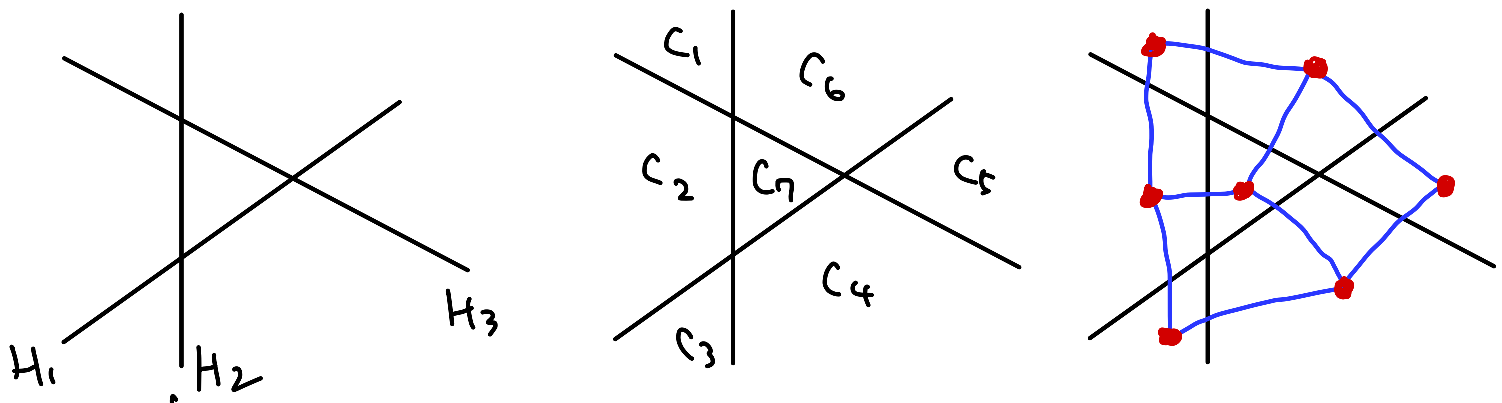
Summary so far:

Setting $\mathcal{A}=\{H_1,\cdots,H_n\}$: central arrangement in $\mathbb{R}^\ell=V$

Notation $M=M(\mathcal{A}):=\mathbb{C}^\ell\setminus\bigcup_{i=1}^n H_i\otimes\mathbb{C}$: complexified complement.

$ch(\mathcal{A}):=\{C : \text{conn. comp (chamber) of } \mathbb{R}^\ell\setminus\bigcup_{i=1}^n H_i\}$.

$\mathcal{T}(\mathcal{A}):=(ch(\mathcal{A}),\{(C,C') : C \text{ and } C' \text{ are adjacent}\})$: adjacency graph. $\overset{\text{equiv}}{\Longleftrightarrow}$ face poset, $\Longleftrightarrow$ Ori. mat.



$\mathcal{A}=\{H_1,H_2,H_3\}$ $\quad ch(\mathcal{A})=\{C_1,C_2,\cdots,C_7\}$ $\quad \mathcal{T}(\mathcal{A})$

- These data determine (homeo. type of) $M(\mathcal{A})$, filtered $\mathcal{VG}(\mathcal{A})$, graded $VG^\bullet(\mathcal{A})$, etc.
- If char $R=2$, $VG^\bullet(\mathcal{A})\cong H^\bullet(M,R)\cong OS^\bullet(\mathcal{A})_R$ (Orlik-Solomon alg.)

$OS^\bullet(\mathcal{A})$ is an invariant of the intersection poset which is much weaker than Ori. mat.

Question: If char $R\neq 2$, how much are $\mathcal{VG}(\mathcal{A})$ and $VG^\bullet(\mathcal{A})$ fine?

3. Main results and Problems.

Question: If $\mathrm{char}\,R \neq 2$, how much are $\mathcal{VG}(\mathcal{A})$ and $VG^{\bullet}(\mathcal{A})$ fine?

<u>Conjecture</u> (Yagi-Y.) $\mathrm{char}\,R \neq 2$.

Let $\mathcal{A}_1, \mathcal{A}_2$ be central arr. in $\mathbb{R}^\ell$. Then TFAE:

(i) Adjacency graphs are isomorphic $T(\mathcal{A}_1) \cong T(\mathcal{A}_2)$

(ii) $\mathcal{VG}(\mathcal{A}_1) \cong \mathcal{VG}(\mathcal{A}_2)$ as filtered algebras.

(iii) $VG^{\bullet}(\mathcal{A}_1) \cong VG^{\bullet}(\mathcal{A}_2)$ as graded algebras.

<u>Rem.</u> (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is easy. (ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (i) are open.

<u>Thm.</u> (Yagi-Y.) $\mathrm{char}\,R \neq 2$. Let $\mathcal{A}_1, \mathcal{A}_2$ be central arr. in $\mathbb{R}^\ell$ which are generic in codim 2.

Then, TFAE:

(i) Adjacency graphs are isomorphic $T(\mathcal{A}_1) \cong T(\mathcal{A}_2)$

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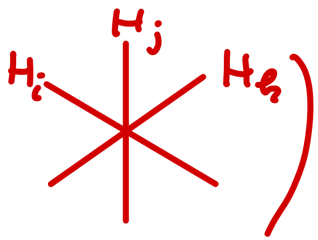
(iii) $VG^{\bullet}(\mathcal{A}_1) \cong VG^{\bullet}(\mathcal{A}_2)$ as graded algebras.

(every $H_i \cap H_j \cap H_k$ has codim $= 3$.

NOT LIKE



)

Today we focus on (ii) $\Rightarrow$ (i). "Reconstruction of $T(\mathcal{A})$ from $\mathcal{VG}(\mathcal{A})$".

3. Main results and Problems.

Thm. (Yagi-Y.) char $R \neq 2$. Let $\mathcal{A}_1, \mathcal{A}_2$ be central arr. in $\mathbb{R}^\ell$ which are generic in codim 2.

Then, TFAE:

(i) Adjacency graphs are isomorphic $\Gamma(\mathcal{A}_1) \cong \Gamma(\mathcal{A}_2)$

(ii) $\mathcal{VG}(\mathcal{A}_1) \cong \mathcal{VG}(\mathcal{A}_2)$ as filtered algebras.

(iii) $VG^\bullet(\mathcal{A}_1) \cong VG^\bullet(\mathcal{A}_2)$ as graded algebras.

Today we focus on (ii) $\Rightarrow$ (i). "Reconstruction of $\Gamma(\mathcal{A})$ from $\mathcal{VG}(\mathcal{A})$".

Step 1 Recovering the vertex set $ch(\mathcal{A})$. (using only char $R \neq 2$).

$$\mathrm{Idem}(\mathcal{VG}(\mathcal{A})) = \{ f \in \mathcal{VG}(\mathcal{A}) \mid f^2 = f \}$$

$$= \{ \sum_{c \in S} 1_c \mid S \subseteq ch(\mathcal{A}) \}.$$

$f \in \mathrm{Idem}(\mathcal{VG}(\mathcal{A}))$ is primitive idempotent

$$\Leftrightarrow f \neq 0, \text{ and } f = \underset{\in \mathrm{Idem}}{f_1} + \underset{\in \mathrm{Idem}}{f_2}, \text{ then } f_1 = 0 \text{ or } f_2 = 0.$$

Prop (char $R \neq 2$) f is prim. idem $\Rightarrow \exists c \in ch(\mathcal{A})$ s.t. $f = 1_c$.

(If char $R = 2$, $(1,1) = (1,0) + (0,1)$, $(0,1) = (1,0) + (1,1)$. NO Primitive idem.)

Hence we have

$$\mathrm{Prim\,Idem}(\mathcal{VG}(\mathcal{A})) = \{ 1_c \mid c \in ch(\mathcal{A}) \}.$$

3. Main results and Problems.

Thm. (Yagi-Y.) char $R \neq 2$. Let $\mathcal{A}_1, \mathcal{A}_2$ be central arr. in $\mathbb{R}^\ell$ which are generic in codim 2.

Then, TFAE: (i) Adjacency graphs are isomorphic $\mathcal{T}(\mathcal{A}_1) \cong \mathcal{T}(\mathcal{A}_2)$

(ii) $\mathcal{VG}(\mathcal{A}_1) \cong \mathcal{VG}(\mathcal{A}_2)$ as filtered algebras.

Today we focus on (ii) $\Rightarrow$ (i). "Reconstruction of $\mathcal{T}(\mathcal{A})$ from $\mathcal{VG}(\mathcal{A})$".

Step 1 Recovering the vertex set $ch(\mathcal{A})$. $\quad \mathrm{Prim\,Idem}(\mathcal{VG}(\mathcal{A})) = \{1_C \mid C \in ch(\mathcal{A})\}$.

Step 2 Recovering the set of Heaviside functions.

$$\mathcal{H}(\mathcal{A}) := \{x_i^+, x_i^- \mid i = 1, \cdots, n\}.$$

Once we obtain $\mathcal{H}(\mathcal{A})$, we can recover adjacency relation by using

$$C \subseteq H_i^+ \Leftrightarrow 1_C \cdot x_i^+ \neq 0 \Leftrightarrow 1_C \cdot (1 - x_i^+) = 0.$$

Clearly, we have $\mathcal{H}(\mathcal{A}) \subseteq F_1\mathcal{VG}(\mathcal{A}) \cap \mathrm{Idem}(\mathcal{A})$. The crucial fact is the following:

Prop (char $R \neq 2$, $\mathcal{A}$: generic in codim 2). $\mathcal{H}(\mathcal{A}) = F_1\mathcal{VG}(\mathcal{A}) \cap \mathrm{Idem}(\mathcal{A}) \setminus \{0, 1\}$.

(proof) Let $f \in F_1 \cap \mathrm{Idem} \setminus \{0, 1\}$.

Recall: $f \in F_1 \Leftrightarrow \rho_{H_i}(f)$ is constant for $\forall\, i = 1, \cdots, n$.

Suppose that $\exists H_i$ s.t. $\rho_{H_i}(f) \neq 0$. Then, ...

3. Main results and Problems.

<u>Step2</u> Recovering the set of Heaviside functions. $\mathcal{H}(\mathcal{A}) := \{x_i^+, x_i^- \mid i=1,\cdots,n\}$.

Clearly, we have $\mathcal{H}(\mathcal{A}) \subseteq F_1\mathcal{VG}(\mathcal{A}) \cap \mathrm{Idem}(\mathcal{A})$. The crucial fact is the following:

<u>Prop</u> (char $R \neq 2$, $\mathcal{A}$: generic in codim 2). $\mathcal{H}(\mathcal{A}) = F_1\mathcal{VG}(\mathcal{A}) \cap \mathrm{Idem}(\mathcal{A}) \setminus \{0,1\}$.

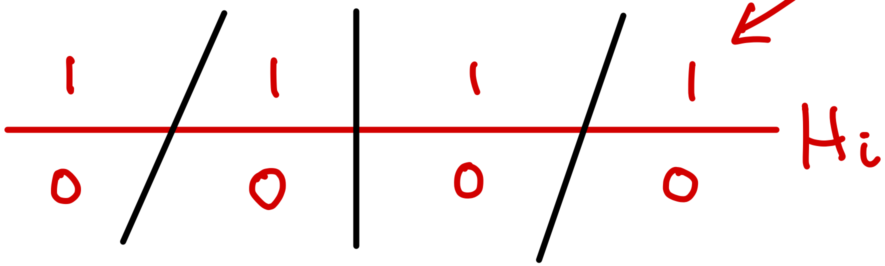
(proof) Let $f \in F_1 \cap \mathrm{Idem} \setminus \{0,1\}$.

Recall: $f \in F_1 \Leftrightarrow \rho_{H_i}(f)$ is constant for $\forall\, i=1,\cdots,n$.

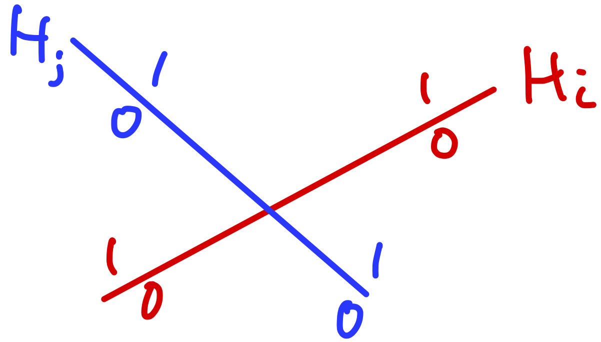
Suppose that $\exists H_i$ s.t. $\rho_{H_i}(f) \neq 0$. Then, $\cdots$

around H_i, it must look like :

$f(c)^2 = f(c)$ implies $f(c) = 0$ or 1.



Suppose that $\exists H_i, H_j$ s.t. $\rho_{H_i}(f) \neq 0$, $\rho_{H_j}(f) \neq 0$. Then,



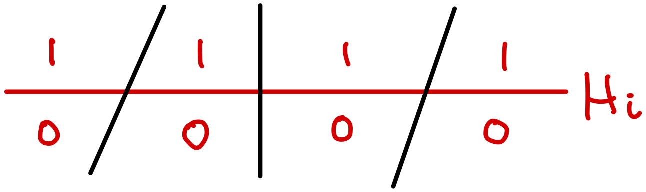
Contradiction! Hence there is at most one H_i s.t. $\rho_{H_i}(f) \neq 0$. $\rightsquigarrow$ 1 or a Heaviside. (Q.E.D.)

3. Main results and Problems.

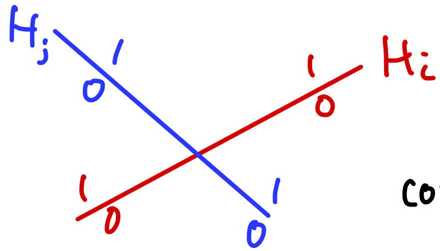
Prop (char $R \neq 2$, A: generic in codim 2). $\mathcal{H}(A) = F_1\mathcal{VG}(A) \cap \mathrm{Idem}(A) \setminus \{0,1\}$.

(proof) Let $f \in F_1 \cap \mathrm{Idem} \setminus \{0,1\}$. Recall: $f \in F_1 \Leftrightarrow \rho_{H_i}(f)$ is constant for $\forall i = 1, \cdots, n$.

Suppose that $\exists H_i$ s.t. $\rho_{H_i}(f) \neq 0$. Then, around H_i, it must look like :



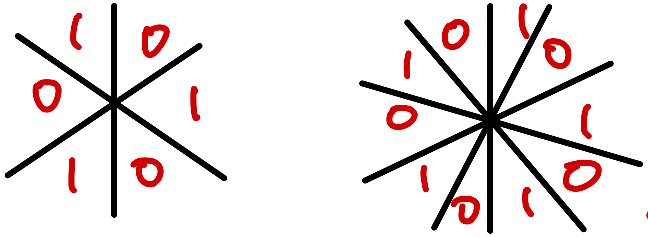
Suppose that $\exists H_i, H_j$ s.t. $\rho_{H_i}(f) \neq 0$, $\rho_{H_j}(f) \neq 0$. Then,



contradiction!

Hence there is at most one H_i s.t. $\rho_{H_i}(f) \neq 0$. $\rightsquigarrow$ 1 or a Heaviside. (Q.E.D.)

Rem. If $\exists\, H_i \cap H_j \cap H_k$ with codim $= 2$, then there are non-Heaviside functions



$\in F_1 \cap \mathrm{Idem}(\mathcal{VG}(A))$

Rem. We can also prove any $f \in F_1 \cap \mathrm{Idem}$ is of this form (or a Heaviside).

3. Main results and Problems.

<u>Thm.</u> (Yagi–Y.) char $R \neq 2$. Let $\mathcal{A}_1, \mathcal{A}_2$ be central arr. in $\mathbb{R}^\ell$ which are generic in codim 2.

Then, TFAE: (i) Adjacency graphs are isomorphic $\Gamma(\mathcal{A}_1) \cong \Gamma(\mathcal{A}_2)$

(ii) $\mathcal{VG}(\mathcal{A}_1) \cong \mathcal{VG}(\mathcal{A}_2)$ as filtered algebras.

(iii) $VG^{\bullet}(\mathcal{A}_1) \cong VG^{\bullet}(\mathcal{A}_2)$ as graded algebras.

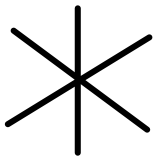
Consider automorphism group of these objects.

$$\mathrm{Aut}_{\mathrm{graph}}(\Gamma(\mathcal{A})) \hookrightarrow \mathrm{Aut}_{\mathrm{filt}}(\mathcal{VG}(\mathcal{A})) \hookrightarrow \mathrm{Aut}_{\mathrm{alg}}(\mathcal{VG}(\mathcal{A})) = \mathrm{Bij}(\mathrm{ch}(\mathcal{A})).$$

<u>Thm.</u> (Y–Y) char $R \neq 2$. $\mathcal{A}$: generic in codim 2, then

$$\mathrm{Aut}_{\mathrm{graph}}(\Gamma(\mathcal{A})) \cong \mathrm{Aut}_{\mathrm{filt}}(\mathcal{VG}(\mathcal{A})).$$

<u>Ex.</u> $\mathcal{A}$:



then $\quad \mathrm{Aut}_{\mathrm{graph}}(\Gamma(\mathcal{A})) \subsetneq \mathrm{Aut}_{\mathrm{filt}}(\mathcal{VG}(\mathcal{A}))$

$\# = 12 \qquad\qquad \# = 48$

<u>Question 2</u> When $\mathrm{Aut}_{\mathrm{graph}}(\Gamma(\mathcal{A})) \cong \mathrm{Aut}_{\mathrm{filt}}(\mathcal{VG}(\mathcal{A}))$?

3. Main results and Problems.

<u>Question 4</u> Suppose $R=\mathbb{F}_2$. Can we recover $\mathcal{T}(A)$ from the filtered alg $\mathcal{VG}(A)$?

Two concrete test cases:

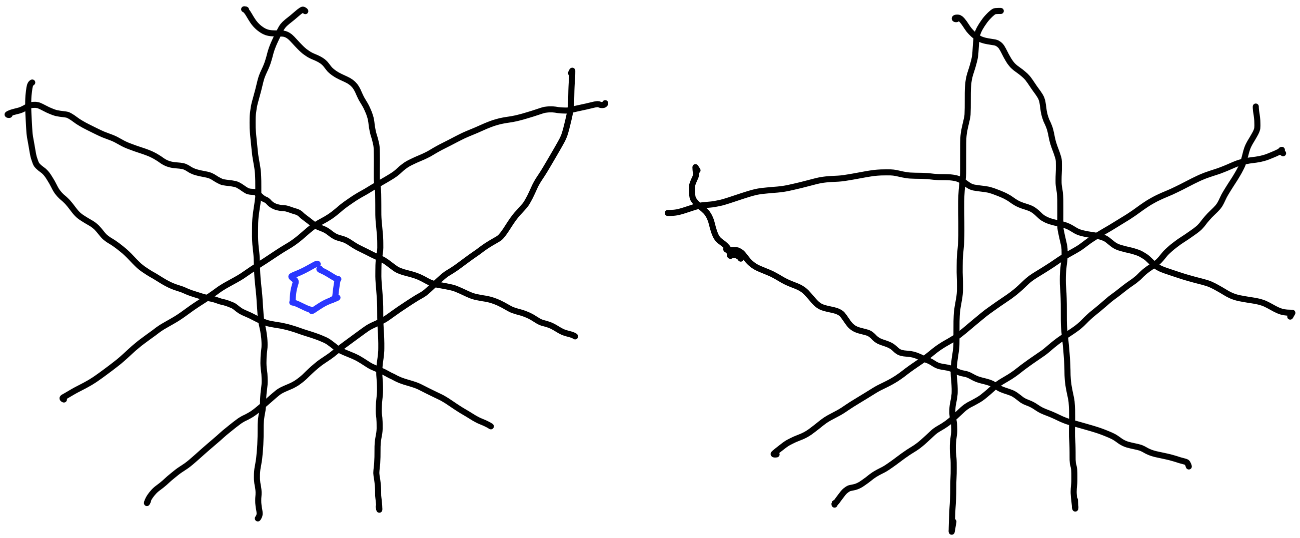
3. Main results and Problems.

<u>Question 4</u> Suppose $R=\mathbb{F}_2$. Can we recover $\mathcal{J}(\mathcal{A})$ from the filtered alg $\mathcal{VG}(\mathcal{A})$?

Two concrete test cases:

(1) generic arrangements: $\mathcal{A}_1, \mathcal{A}_2$: generic 6 planes in $\mathbb{R}^3$

$\mathcal{A}_1$ $\qquad\qquad$ $\mathcal{A}_2$



NO Hexagons

- If char $R \neq 2$, $\mathcal{VG}(\mathcal{A}_1) \not\cong \mathcal{VG}(\mathcal{A}_2)$.
- If $R=\mathbb{F}_2$, $VG^\bullet(\mathcal{A}_1) \cong VG^\bullet(\mathcal{A}_2)$ (because intersection lattices are isom. $OS^\bullet(\mathcal{A}_1) \cong OS^\bullet(\mathcal{A}_2)$)

Question: If $R=\mathbb{F}_2$, $\mathcal{VG}(\mathcal{A}_1) \underset{?}{\cong} \mathcal{VG}(\mathcal{A}_2)$ as filtered algebras?

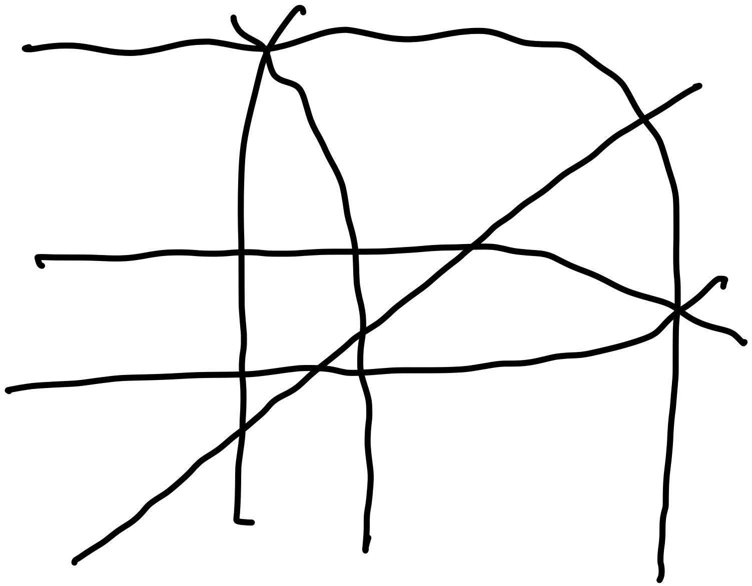
3. Main results and Problems.

<u>Question 4</u> Suppose $R=\mathbb{F}_2$. Can we recover $\mathcal{J}(\mathcal{A})$ from the filtered alg $\mathcal{VG}(\mathcal{A})$?

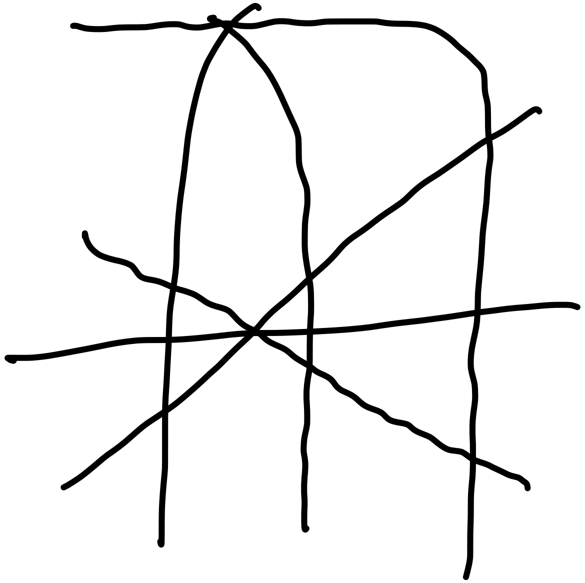
Two concrete test cases:

(2) Non isom. intersection lattice.

$\mathcal{A}_1$



$\mathcal{A}_2$



- Even if char $R \neq 2$, we do not know whether $\mathcal{VG}(\mathcal{A}_1) \cong \mathcal{VG}(\mathcal{A}_2)$ or not.
- $OS^{\bullet}(\mathcal{A}_1) \ncong OS^{\bullet}(\mathcal{A}_2)$ ($M(\mathcal{A}_1)$ and $M(\mathcal{A}_2)$ are homotopy equiv. but not homeomorphic)
- In particular, if $R=\mathbb{F}_2$, $VG^{\bullet}(\mathcal{A}_1) \ncong VG^{\bullet}(\mathcal{A}_2)$.

Question: If $R=\mathbb{F}_2$, $\mathcal{VG}(\mathcal{A}_1) \underset{?}{\cong} \mathcal{VG}(\mathcal{A}_2)$ as filtered algebras?

From Yagi-Y.
Fig 1.