Topological invariants of combinatorial line arrangements

Sakumi Sugawara (Hokkaido)

AGATHA, in Krakow
2026 June 5

Contents

§1 Topology of arrangements
§2 Combinatorial line arrangements
§3 Main results

Ref.

[1]: arXiv:2507.06728
[2]: In preparation.

- $H^*(U) \cong A^\bullet(\mathcal{A})$
- $H^*(\partial U) \cong D(H^*(U))$
- Minimality of U.

⇝ Generalize? Extend?

Abstract: Generalizing well-known topological results to combinatorial line arr. or its topological realization.

§1 Topology of arrangements

Reviewing what and how topological invariants are combinatorially described.

§1 Preliminaries.

- $\mathcal{A} = \{H_0, \dots, H_n\}$: a hyperplane arrangement in $\mathbb{CP}^\ell$
- $U(\mathcal{A}) := \mathbb{CP}^\ell \setminus \bigcup_{i=0}^{n} H_i$: the complement
- $\partial U(\mathcal{A}) := \partial_U\left(\bigcup_{i=0}^{n} H_i\right)$: the boundary manifold
- $L(\mathcal{A}) := \left\{ \bigcap_{H \in \mathcal{B}} H \neq \phi \mid \mathcal{B} \subseteq \mathcal{A} \right\}$: the intersection poset

Q. What topological information is determined by $L(\mathcal{A})$?

- Yes : $b_*(U)$, $H^*(U)$, $H^*(\partial U)$, homeo type of ∂U $(\ell=2)$, ...
- No : $\pi_1(U)$, $\pi_1(\partial U)$ $(\ell \geq 3)$, ...
- There are many open problems. (higher homotopy, Milnor fiber, ...)

§1 Combinatorial description of $b_*(U)$.

Def · The characteristic polynomial $\chi(c\mathcal{A},t)$ (of coning) is defined by

$$\chi(c\mathcal{A},t) = \sum_{I\subset\{0,\dots,n\}} (-1)^{|I|} t^{\dim cH_I}$$

$\left(\begin{array}{l} cH_I = \text{coning of } \bigcap_{i\in I} H_i \\ c\emptyset := \{0\} \end{array} \right)$

· The reduced char. poly. is defined by

$$\bar{\chi}(c\mathcal{A},t) = \frac{\chi(c\mathcal{A},t)}{t-1}$$

They are determined by $L(\mathcal{A})$.

Thm (Orlik-Solomon)

$$\sum_{k=0}^{\ell} (-1)^{\ell-k} b_{\ell-k}(U(\mathcal{A}))\, t^k = \bar{\chi}(c\mathcal{A},t)$$

§1 Combinatorial description of $b_*(U)$.

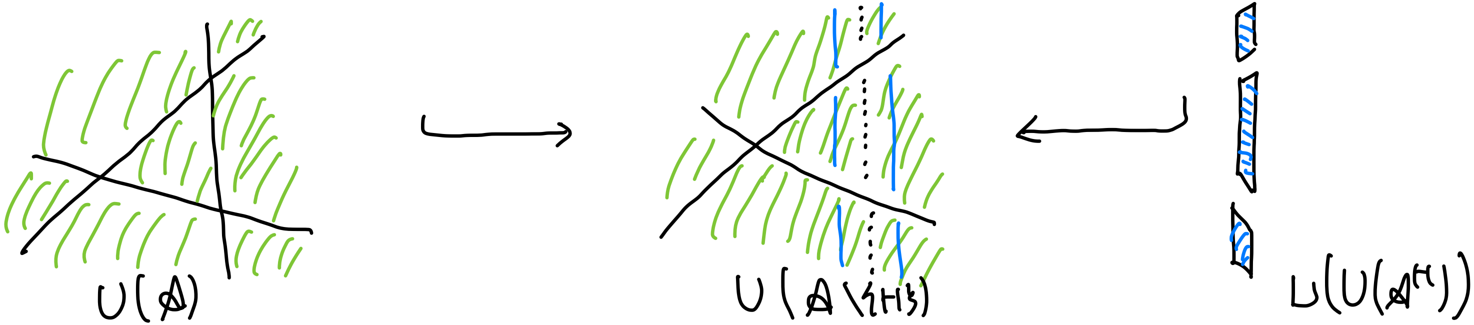
$$\sum_{k=0}^{\ell}(-1)^{\ell-k} b_{\ell-k}(U(\mathcal{A}))\, t^k = \bar{\chi}(c\mathcal{A}, t)$$

Sketch of proof: Characteristic polynomial satisfies the deletion-restriction formula:

$$\chi(\mathcal{A},t) = \chi(\mathcal{A}\setminus\{H\},t) - \chi(\mathcal{A}^H,t)$$

This agrees with the splitting Mayer-Vietoris sequence:

$$0 \to H_*(U(\mathcal{A}^H)\times D^*) \to \underbrace{H_*(U(\mathcal{A}))}_{} \oplus \underbrace{H_*(U(\mathcal{A}^H)\times D^2)}_{} \to H_*(U(\mathcal{A}\setminus H)) \to 0$$



□

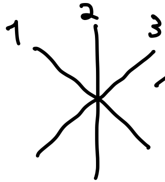
4/29

§1 Combinatorial description of $\underline{\underline{H^*(U)}}$.

Def The Orlik-Solomon algebra $A^*(c\mathcal{A})$ is defined by

- $A^*(c\mathcal{A}) = \dfrac{\Lambda(\mathbb{Z}e_0 \oplus \cdots \oplus \mathbb{Z}e_n)}{\langle \partial e_S \mid S \subset \{0,\ldots,n\} \text{ is dependent} \rangle}$ (gr. $\mathbb{Z}$-alg.)
- $\bar{A}^*(c\mathcal{A}) = A^*(c\mathcal{A}) \Big/ \sum_{i=0}^{n} e_i$: reduced OS alg.

Rmk Each $A^k(c\mathcal{A})$ is a free $\mathbb{Z}$-module with a "nbc-basis".



$e_1e_2 + e_2e_3 + e_3e_1 = 0$

$\Leftrightarrow e_2e_3 = \underbrace{e_1e_2 - e_1e_3}_{\text{nbc}}$

Thm (Orlik-Solomon '80)

$H^*(U(\mathcal{A});\mathbb{Z}) \cong \bar{A}^*(c\mathcal{A})$ as gr. $\mathbb{Z}$-alg.s

§1 Combinatorial description of $H^*(U)$.

Thm (Orlik-Solomon '80)

$H^*(U(\mathcal{A});\mathbb{Z}) \cong \bar{A}^*(c\mathcal{A})$ as gr. $\mathbb{Z}$-alg.s

Sketch of (famous) proof

$$H^*(U(\mathcal{A})) \underset{\substack{\uparrow \\ \text{Brieskorn}}}{\cong} \underset{\textcircled{!}}{\Omega^*\left\langle \frac{1}{2\pi\sqrt{-1}}\frac{d\alpha_i}{\alpha_i} \right\rangle} \underset{\substack{\uparrow \\ \text{OS}}}{\cong} A^*(c\mathcal{A})$$

□

Rmk

- In this proof, the existence of defining equation is essential.
- There is another, more combinatorial proof for $H^*(U) \cong A^*(\mathcal{A})$ by Björner-Ziegler, which is valid for "2-pseudo arrangements". Their method uses a stratifaction on $S^{2\ell-1} \cap \mathcal{A}$ and Alexander dual.

§1 Combinatorial description of $H^*(\partial U)$.

6/29

Recall : $\partial U = \partial \nu(\bigcup_{i=0}^{n} H_i)$: the boundary mfd, cpt $(2\ell-1)$-dim. mfd.

Def · $A = \bigoplus_{i=0}^{k} A^i$: gr. $\mathbb{Z}$-alg. The double $D(A)$ is defined by

- $D(A)^i = A^i \oplus \mathrm{Hom}(A^{2k+1-i}, \mathbb{Z})$ (underlying module)
- $(a,f)\cdot(b,g) := (ab, ag+fb)$ (multiplication)

	D(A)
5	$\bar{A}^0$
4	$\bar{A}^1$
3	$A^3 \bar{A}^2$
2	$A^2 \bar{A}^3$
1	A^1
0	A^0

Rmk · $D(A)$ is a P.D. alg. of dim $(2k+1)$.

· $D(A)$ satisfies split ex. seq. $0 \to \mathrm{Hom}(A,\mathbb{Z}) \to D(A) \to A \to 0$
(trivial extension)

Thm (Cohen-Suciu)

$$H^*(\partial U; \mathbb{Z}) \cong D(H^*(U;\mathbb{Z})) \; (\cong D(\bar{A}^*(c\mathcal{A})))$$

as gr. $\mathbb{Z}$-alg.s

§1 Combinatorial description of $H^*(\partial U)$.

Thm (Cohen–Suciu)

$$H^*(\partial U;\mathbb{Z}) \cong D(H^*(U;\mathbb{Z})) \ \left(\cong D(\overline{A}^*(c\mathcal{A}))\right)$$

as gr. $\mathbb{Z}$-alg.s

Sketch of proof

We have a short ex. seq. as modules

$$0 \to H^k(U) \to H^k(\partial U) \to \overset{\overset{H_{2\ell+1-k}(U)}{\wr\wr \ \text{Lefschetz dual}}}{H^{k+1}(U,\partial U)} \to 0$$

and it splits as an algebra by Hodge theory.

Rmk · The existence of U is essential.

· When $\ell=2$, ∂U is constructed from $L(\mathcal{A})$ by "plumbing".

§1 Summary of §1

Topological invariants	Combinatorial objects	Proof methods
$b_*(U)$	$\bar{\chi}(c\mathcal{A}, t)$	· Deletion-restriction & Mayer-Vietoris
$H^*(U)$	$\bar{A}^*(c\mathcal{A})$	· $\Omega\left\langle \frac{1}{2\pi\sqrt{-1}} \frac{d\alpha_i}{\alpha_i} \right\rangle$ (Stratification & Alexander dual)
$H^*(\partial U)$	$D(\bar{A}(c\mathcal{A}))$	Exact seq. of $(U, \partial U)$.

Q: Can we extend these formulas to generalized objects?

(hyperplane arrangements "$\subsetneq$" Matroids) $\rightsquigarrow$ Topology?

§2 Combinatorial line arrangements

§2 To what we generalize these formulas?

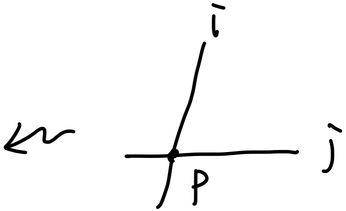
9/29

Def · $\mathcal{I}=\{0,1,\dots,n\}$: finite set, $\mathcal{P}\subset 2^{\mathcal{I}}$.

· A pair $\mathcal{C}=(\mathcal{I},\mathcal{P})$ is called a combinatorial line arrangement

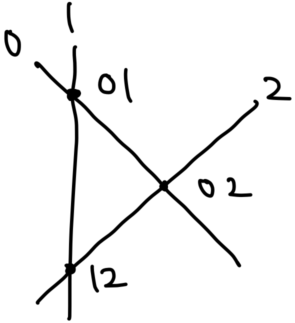
$\Leftrightarrow$ i) $\forall p\in\mathcal{P}$ $|p|\geq 2$

ii) $\forall i,j\in\mathcal{I}$ $\exists!\, p\in\mathcal{P}$ s.t. $i,j\in p$.



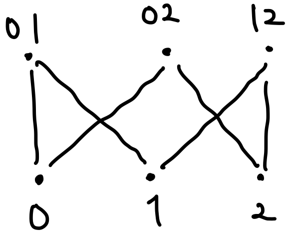
· $\mathcal{C}=(\mathcal{I},\mathcal{P})$ defines a poset $L(\mathcal{C})=\mathcal{I}\sqcup\mathcal{P}$ $(i<p \Leftrightarrow i\in p)$.

Ex.



· $\mathcal{I}=\{0,1,2\}$

· $\mathcal{P}=\{01,02,12\}$



$L(\mathcal{C})$

lattice of flats (proper part)

§2 Example of combinatorial line arrangements

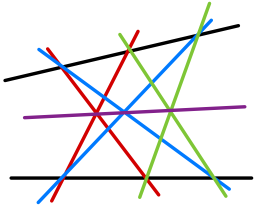
- $\mathcal{C} = (\mathcal{L}, \mathcal{P})$: comb. line arr. · $L(\mathcal{C})$: its poset.

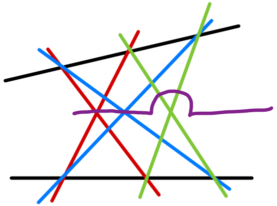
Ex. · $\mathcal{A}$: (classical) line arr. in $\mathbb{C}P^2$. (any field is ok)

$\Rightarrow$ $\mathcal{C} = (\mathcal{A}, \mathrm{Sing}(\mathcal{A}))$ is a comb. line arr., and

$L(\mathcal{C}) \cong L(\mathcal{A})$.

· Non-Pappus comb. line arr. ⇝ Non-realizable comb. line arr.





Fact (Non-trivial) $\{$Comb. line arr.s$\}$ $\xleftrightarrow{1:1}$ $\{$Simple matroids of rank 3$\}$

⇝ $\chi(\mathcal{C}, t)$ and $A^*(\mathcal{C})$ are defined similarly for any $\mathcal{C}$.

11/29

§2 Topological Realization

Def $\mathcal{C} = (\mathcal{L}, \mathcal{P})$: comb. line arr.

$\mathcal{A} = \{H_i\}_{i \in \mathcal{L}}$ is called a topological realization of $\mathcal{C}$

$\iff$ · Each H_i is a locally-flat emb'd $S^2 \hookrightarrow \mathbb{CP}^2$, and

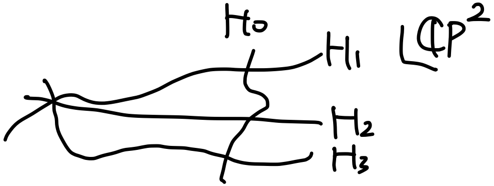
$[H_i] = [\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})$.

· Each line intersects transversally as $\mathcal{P}$.

- Let $U(\mathcal{A}) = \mathbb{CP}^2 \setminus \bigcup_{i \in \mathcal{L}} H_i$ be the complement.
- Similarly, we can define smooth, symplectic, algebraic realizations.

Ex. $\mathcal{L} = \{0, 1, 2, 3\}$

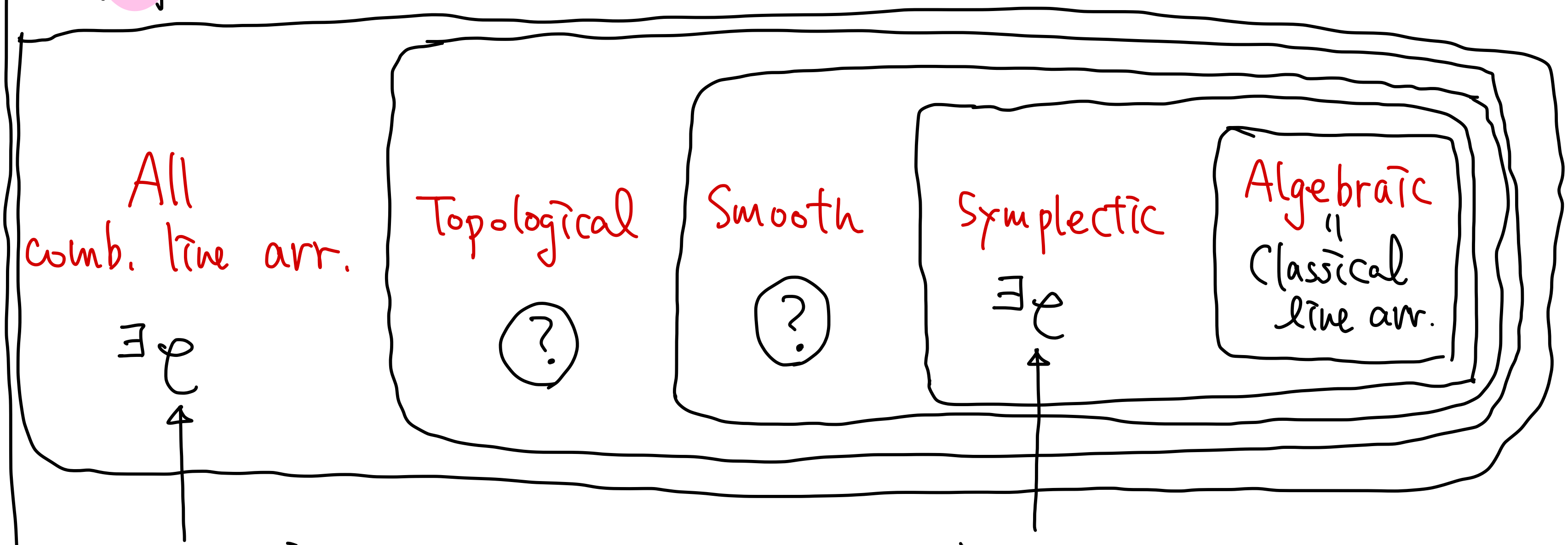
$\mathcal{P} = \{01, 02, 03, 123\}$



12/29

§2 Remarks on realizability

Prop. (Ruberman–Starkston)



Combinatorics of $\mathbb{F}_q P^2$, etc.

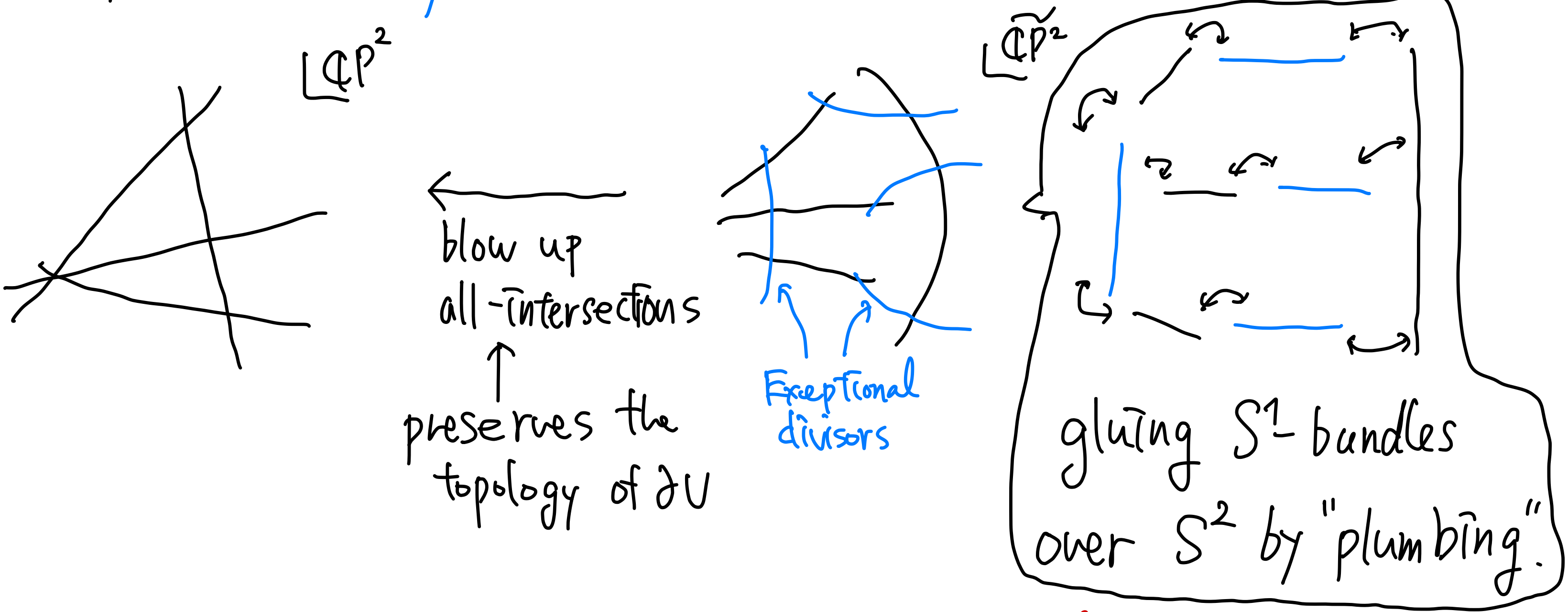
Non-Pappus

(generally, non-realizable oriented matroids live here.)

§2 The boundary manifold for topological line arr.

Suppose $\mathcal{C} = (\mathcal{L}, \mathcal{P})$ is topologically realizable.

Then, the boundary manifold is constructed as follows:



blow up all-intersections

preserves the topology of ∂U

gluing S^1-bundles over S^2 by "plumbing".

This procedure can be done without realization!

What we need is only $L(\mathcal{A})$!

§2 The boundary manifold for ANY comb. line arr.

Let $\mathcal{C} = (\mathcal{L}, \mathcal{P})$ be (any) comb. line arr.

The poset $L(\mathcal{C}) = \mathcal{L} \sqcup \mathcal{P}$ can be considered as a graph.

① Equip an S^1-bundle $E(x)$ over S^2 for each $x \in L(\mathcal{C})$

② Glue $E(i)$ and $E(p)$ by plumbing if $i \in P$.

→ the resulting manifold $\partial U(\mathcal{C})$ is an ori. closed 3-mfd,

$\partial U(\mathcal{C}) \approx \partial U(\mathcal{A})$ if $\mathcal{C}$ is realizable. (The "boundary mfd" for $\mathcal{C}$).

Fact (Ruberman-Starkston)

$\mathcal{C}$ is symplectic realizable $\Longleftrightarrow$ $\partial U(\mathcal{C})$ is strongly symplectic fillable (with a certain contact str.)

§2 Summary of §2

(Just a matroid of rank 3)

$$\begin{tikzcd} \bullet\ \mathcal{A}\text{: line arr. in } \mathbb{C}P^2 \arrow[r, squiggly, "\text{generalize}"] \arrow[dr, squiggly] & \mathcal{C}=(\mathcal{L},\mathcal{P})\text{ : comb. line arr.} \arrow[d, squiggly] \\ & \mathcal{A}\text{: topological realization} \end{tikzcd}$$

- $U(\mathcal{A})$ is not determined by $L(\mathcal{A})$, but $\partial U(\mathcal{A})$ is determined.

  ⇝ $\partial U(\mathcal{C})$ is defined for ANY comb. line arr. $\mathcal{C}$.

Q. What about topological invariants?

⟶ Main results.

§3 Main results

- Combinatorial description
- On the minimality.

§3 Main result I.

$\mathcal{C}=(\mathcal{L},\mathcal{P})$: conn. line arr. $\mathcal{A}$: topological realization of $\mathcal{C}$.

Thm ((1),(2): [2], (3): [1])

1) $\sum_{i=0}^{2}(-1)^{2-i}b_{2-i}(U(\mathcal{A}))t^{i}=\bar{\chi}(\mathcal{C},t)$

2) $H^*(U(\mathcal{A});\mathbb{Z})\cong\bar{A}^*(\mathcal{C})$

3) $H^*(\partial U(\mathcal{C});\mathbb{Z})\cong D(\bar{A}^*(\mathcal{C}))$

Rmk The classical results hold simliarly for the above cases, however, we CANNOT use the classical method for the proof.

§3 Betti numbers : similarly proven

Thm (1) $\sum_{i=0}^{2}(-1)^{2-i} b_{2-i}(U(\mathcal{A}))t^{i} = \bar{\chi}(e,t)$

Sketch of proof

- The deletion-restriction formula also holds for e :

$$\bar{\chi}(e,t) = \bar{\chi}(\underset{\substack{\uparrow \\ \text{the deletion}}}{e'},t) - \bar{\chi}(\underset{\substack{\uparrow \\ \text{the restriction} \\ \text{(not a comb. line arr.)}}}{e''},t)$$

- We have a splitting MV seq, similarly.

$$0 \to H_*(U(\mathcal{A}^H)\times D^*) \to H_*(U(\mathcal{A}^H)\times D^2)\oplus H_*(U(\mathcal{A})) \to H_*(U(\mathcal{A}\setminus H)) \to 0$$

⇝ For b_*, the classical proof also holds ?

§3 Cohomology ring : What is the difference?

Thm (2) $H^*(U(\mathcal{A});\mathbb{Z}) \cong \bar{A}^*(\mathcal{C})$

Recall: Classical proof needs the defining equation to define "$\frac{d\alpha_i}{\alpha_i}$".

- But, we CANNOT hope the definining equation for topological realization in general.
- Also, Björner-Ziegler's method does not apply to topological realization, because we cannot hope a nice stratification.

$\rightsquigarrow$ We need a NEW method which is valid for topological line arrangements!

§3 Cohomology ring : A homological method

Thm (2) $H^*(U(\mathcal{A});\mathbb{Z}) \cong \bar{A}^*(\mathcal{C})$

Instead of computing the cup product $\cup : H^k(U)\times H^\ell(U)\to H^{k+\ell}(U)$, we will compute the homology intersection product

$$\cdot \quad H_{4-k}(U,\partial U)\times H_{4-\ell}(U,\partial U)\longrightarrow H_{4-k-\ell}(U,\partial U)$$

which is defined by Poincaré–Lefschetz duality. ($\dim_{\mathbb{R}} U = 4$)

In particular, the non-trivial product is only

$$H_3(U,\partial U)\times H_3(U,\partial U)\longrightarrow H_2(U,\partial U).$$

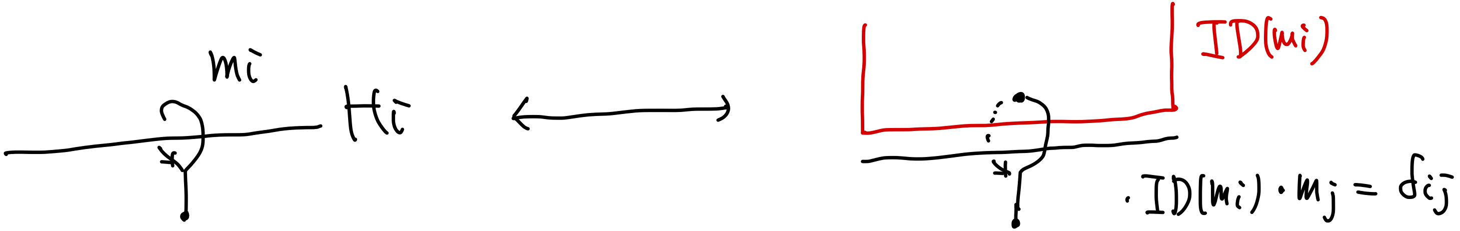
(dual of $H^1\times H^1\to H^2$)

→ We will construct a basis for $H_*(U,\partial U)$ to compute this.

§3 Cohomology ring: Basis of $H_3(U,\partial U)$

Thm (2) $H^*(U(\mathcal{A});\mathbb{Z}) \cong \bar{A}^*(\mathcal{C})$

Basis of $H_3(U,\partial U)$ is constructed by the intersection dual.



$\cdot\ ID(m_i)\cdot m_j = \delta_{ij}$

$\{ID(m_1),\dots,ID(m_n)\}$ is a basis of $H_3(U,\partial U)$.

→ It is enough to construct a 3-mfd F_i w/ ∂. s.t.

- $F_i \cdot m_j = \delta_{ij}$ (m_j: meridian of H_j)
- $F_i \cap \partial U = \partial F_i$ $\longrightarrow$ $[F_i] = ID(m_i)$!

§ 3 Cohomology ring : Construction the Intersection dual.

21/29

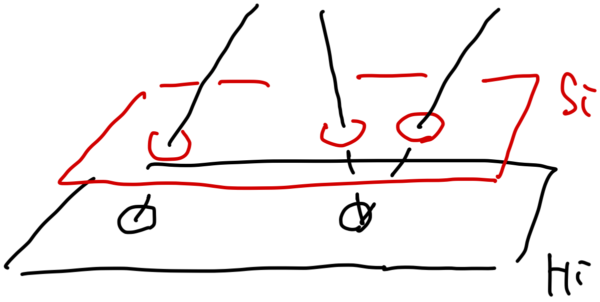
Thm (2) $H^*(U(\mathcal{A});\mathbb{Z}) \cong \overline{A}^*(\mathcal{C})$

Intersection dual F_i :
- $F_i \cdot m_j = \delta_{ij}$
- $F_i \cap \partial U = \partial F_i$

Firstly, construct the boundary ∂F_i.

Locally around H_i, ∂U looks like S^1-b'dle over a surface.

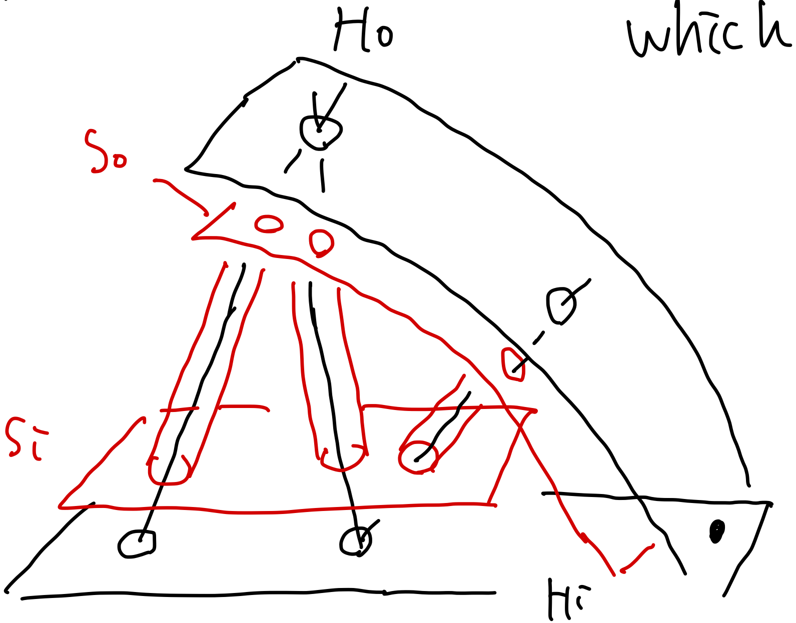
→ Take a section S_i. Then, $S_i \cdot m_j = \delta_{ij}$



§ 3 Cohomology ring : Construction the intersection dual.

Thm (2) $H^*(U(\mathcal{A}); \mathbb{Z}) \cong \bar{A}^*(\mathcal{C})$

- Apply the same construction for S_0, and connect S_i & S_0 by annuli. Then, we get a closed surface, which will become ∂F_i.



§3 Cohomology ring: Construction the intersection dual.

Thm (2) $H^*(U(\mathcal{A});\mathbb{Z}) \cong \bar{A}^*(\mathcal{C})$

- Finally, capping ∂F_i by a pencil-like method.

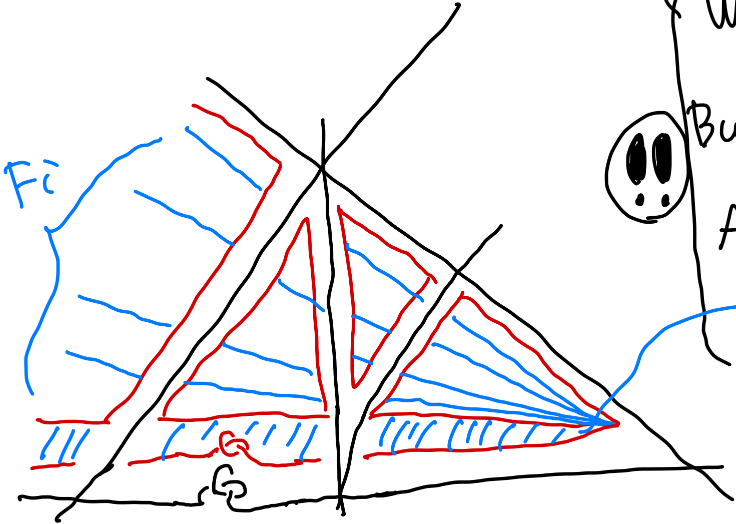
($\gamma : \mathbb{C}P^2 \dashrightarrow \mathbb{C}P^1$, the fiber of a path $[1:0] \to [0:1]$ in $\mathbb{C}P^1$.)

$$[x:y:z] \mapsto [\alpha_i : \alpha_0]$$

We can't hope the def. eq. for H_i.

But, H_i is homologous to a straight line.

After "unknotting operation" to H_i, we apply pencil method.



§ 3 The doubling formula $H^*(\partial U) \cong D(\bar{A}^*(\mathcal{C}))$

Thm (3) $H^*(\partial U(\mathcal{C}); \mathbb{Z}) \cong D(\bar{A}^*(\mathcal{C}))$

The proof due to Cohen-Suciu uses the exact seq $(U, \partial U)$.

But for general $\mathcal{C}$, we cannot hope the existence of $\underline{\underline{U}}$.

So, we will compute $H^*(\partial U(\mathcal{C}); \mathbb{Z})$ directly,

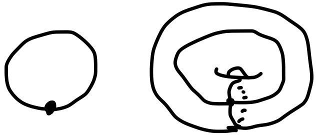
and prove it is isomorphic to $D(\bar{A}^*(\mathcal{C}))$.

Here, we also took homological method and computed

$$H_2(\partial U) \times H_2(\partial U) \to H_1(\partial U).$$

§ 3 Minimality

Besides, there is a property of topological realization which does not share with classical result.



Def X: CW cpx.

X is minimal $\Longleftrightarrow$ $b_k(X) = \#\{k\text{-cells of } X\}$

Ex. · S^1, T^n admits minimal str.

· $\mathbb{RP}^2$ never be minimal. ($H_1(\mathbb{RP}^2) = \mathbb{Z}/2$, $b_1(\mathbb{RP}^2) = 0$)

Thm (Dimca-Papadima, Randell)

$\mathcal{A}$: hyperplane arr in $\mathbb{CP}^\ell$

$U(\mathcal{A}) \underset{\text{h.e.}}{\simeq}$ (minimal CW cpx.)

§3 Main result II

Thm (Dimca–Papadima, Randell)

$\mathcal{A}$: hyperplane arr in $\mathbb{CP}^{\ell}$

$U(\mathcal{A}) \underset{\text{h.e.}}{\simeq}$ (minimal CW cpx.)

Thm [2]

$\mathcal{A}$: symplectic realization of a conic line arr. $\mathcal{C}$

Then, $U(\mathcal{A}) \simeq$ (minimal CW cpx.)

Thm [2]

$^{\exists}\mathcal{C}$, $^{\exists}\mathcal{A}$: its topological realization s.t. $U(\mathcal{A}) \not\simeq$ (minimal cpx)

§3 Minimality of symplectic realization

Thm [2]

$\mathcal{A}$: symplectic realization of a comb line arr. $\mathcal{C}$

Then, $U(\mathcal{A}) \simeq$ (minimal CW cpx.)

Outline of proof

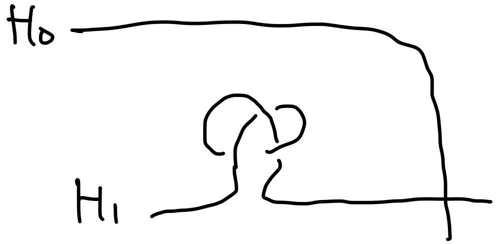
- There is a braid monodromy technique for symplectic curves in $\mathbb{CP}^2$. (Kharlamov-Kulikov)
- (Generalized) Libgober's result states the pres. of π_1 by braid monodromy gives a cell-decomp. of the complement.
- In arrangement case, this gives a minimal CW cpx.

§ 3 Non-minimal example

Thm [2]

$\exists \mathcal{C}$, $\exists \mathcal{A}$: its topological realization s.t. $U(\mathcal{A}) \not\cong$ (minimal cpx)

Proof



Let $\mathcal{C} = \{0,1\}$.

- $H_0 := \mathbb{CP}^1 \subset \mathbb{CP}^2$
- Define H_1 so that it is a "knotted" disk in $\underset{\cong B^4}{\mathbb{CP}^2 \setminus \nu(H_0)}$.
  (spun trefoil)

Then, $U(\mathcal{A}) = \mathbb{CP}^2 \setminus (H_0 \cup H_1) \cong B^4 \setminus (\underbrace{H_1 \setminus (H_1 \cap H_0)}_{\text{knotted 2-disk.}})$

$\rightsquigarrow \pi_1(U(\mathcal{A})) \cong \pi_1(S^3 \setminus (\text{trefoil}))$

$\rightsquigarrow \operatorname{rank}(\pi_1(U(\mathcal{A}))) > 1$. This contradicts to minimality.

§ 3 More non-minimal example ?

Thm [2]

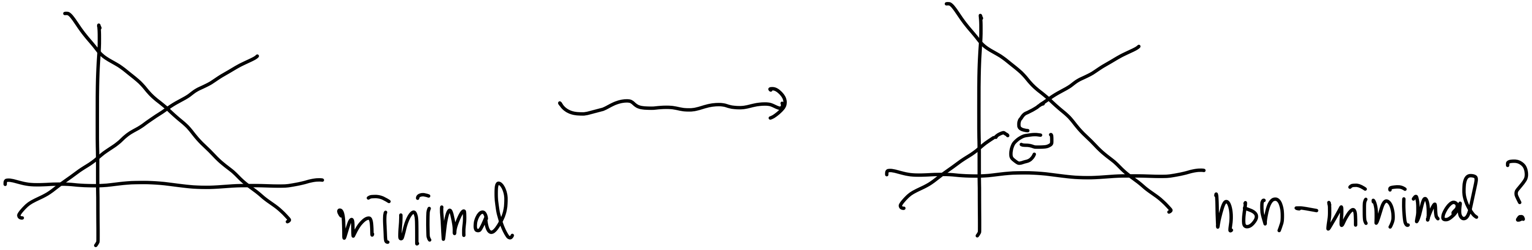
$\exists \mathcal{C}$, $\exists \mathcal{A}$: its topological realization s.t. $U(\mathcal{A}) \not\simeq$ (minimal cpx)

Question

$\mathcal{C}$: topologically realizable $\overset{?}{\Longrightarrow}$ $\exists \mathcal{A}$: its realization s.t. $U(\mathcal{A}) \not\simeq$ (minimal cpx.)

We hope that similar "knotted" construction holds for any arr.

But now, I could not prove it...



Thank you for listening!

[Summary]

- $\mathcal{A}$: line arr in $\mathbb{CP}^2$ $\xrightarrow[\text{generize}]{}$ Topological realizations
- $b_*(U)$ has combinatorial description with similar proof.
- $H^*(U)$ ———————— with a NEW method.
- U is minimal for symplectic arr, but in general, U fails the minimality.