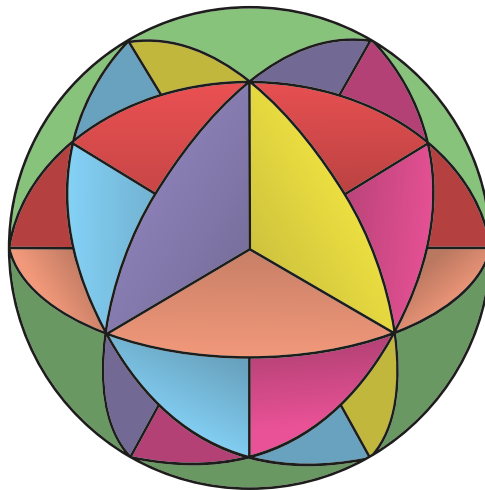


# Open problems booklet

## AGATHA – Algebraic and Geometric Aspects of the Theory of Hyperplane Arrangements 2026

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# Contents

1: Does something on $D^p(\mathcal{A})$ imply that of $D(\mathcal{A})$ ? . . . . .	4
2: Classification of free multiplicities on braid arrangements . . . . .	8
3: A systematic study of matroid orientations using SAT solvers . . . . .	12
4: Projective dimension and Zariski-open subsets of the realization space . . . . .	15
5: Problems of Buildingset arrangements . . . . .	18
6: Torsion in first homology group of the Milnor fiber boundary of arrangements	20
7: The minimum period of the equivariant characteristic quasi-polynomials . . . . .	22
8: Reconstruction of the oriented matroid from the Varchenko-Gelfand Algebra . . . . .	25
9: Enumerating chambers in $m = 1$ Grasstopes . . . . .	27

# 1: Does something on $D^p(\mathcal{A})$ imply that of $D(\mathcal{A})$ ?

(suggested by Takuro Abe)

I am interested in  $D^p(\mathcal{A})$ , a higher order logarithmic derivation module. The definition of it is as follows:

$$D^p(\mathcal{A}) := \{\theta \in \wedge^p \text{Der} S \mid \theta(\alpha_H, f_2, \dots, f_p) \in S\alpha_H \ (\forall H \in \mathcal{A}, \forall f_i \in S)\}.$$

Here  $S$  is a coordinate ring of  $V$ ,  $\mathcal{A}$  is an arrangement in  $V$ ,  $\alpha_H$  is a defining linear form of  $H \in \mathcal{A}$ , and  $\text{Der} S := \bigoplus_{i=1}^{\ell} S\partial_{x_i}$  for the basis  $x_1, \dots, x_{\ell}$  for  $V^*$ . That is not so famous and popular, because (1) it is super hard to compute, at least by hand, and (2) there are almost no place used. For (2), probably the only place which they are used essentially with importance is the Solomon-Terao formula asserting that

$$\lim_{x \rightarrow 1} \text{Hilb}(D^p(\mathcal{A}); x)(t(x-1) - 1)^p = (-1)^{\ell} \chi(\mathcal{A}; t).$$

Recently there are researchers based on this like Solomon-Terao algebras, superspace coinvariants and so on, but still very few. For me, however, the algebraic structure of  $D^p(\mathcal{A})$  itself seems to be interesting. For example, we know that

- (1)  $D^0(\mathcal{A}) \simeq S$  and  $D^{\ell}(\mathcal{A}) \simeq S[-|\mathcal{A}|]$ .
- (2)  $D^p(\mathcal{A})$  is free if  $\mathcal{A}$  is free, and in that case,  $\wedge^p D(\mathcal{A}) = D^p(\mathcal{A})$ . But it is not true in general, e.g., a generic arrangement consisting of four planes in the three dimensional Eucliden space does not satisfy this for  $p = 2$  (Solomon-Terao, 1986).
- (3)  $D^p(\mathcal{A})_x = D^p(\mathcal{A}_X)_x$  for  $X \in L(\mathcal{A})$  and a generic point  $x \in X$ , i.e.,  $D^p$  is a local functor (Solomon-Terao, 1986).
- (4) If  $\text{pd}_S D(\mathcal{A}) = 1$  and  $\mathcal{A}$  is locally free, then  $\wedge^p D(\mathcal{A}) \simeq D^p(\mathcal{A})$  for  $0 \leq p \leq \ell - 2$ . In this case,  $\text{pd}_S D^p(\mathcal{A}) = p$  for  $0 \leq p \leq \ell - 2$ ,  $\text{pd}_S D^{\ell-1}(\mathcal{A}) = \ell - 2$  and  $\text{pd}_S D^{\ell}(\mathcal{A}) = 0$  (Mustata-Schenck, 2001).
- (5) There is an identification

$$D^p(\mathcal{A}) \simeq Q(\mathcal{A})\Omega^{\ell-p}(\mathcal{A})$$

for  $p = 0, \dots, \ell$  (forklore?).

From the viewpoint of freeness, the most fundamental problem is the following.

**Problem 1.** Let  $2 \leq p \leq \ell - 2$ . Then is  $\mathcal{A}$  free if  $D^p(\mathcal{A})$  is free?

As in (2), the converse of Problem 1 is true, and the freeness of  $D^0(\mathcal{A})$  and  $D^{\ell}(\mathcal{A})$  have no information as in (1). So Problem 1 be comes a problem only when  $2 \leq p \leq \ell - 2$  by (5). However, we can show the following easily.

**Theorem 1.**  $\mathcal{A}$  is free if  $D^2(\mathcal{A})$  or  $D^{\ell-2}(\mathcal{A})$  is free.

So the first case we should analyse is when  $\ell = 6$ , and  $D^3(\mathcal{A})$  is free. Does these conditions imply the freeness of  $\mathcal{A}$  or not?

Also, I am interested in several properties which are implied by freeness. The main topic now is the formality. It is known that  $\mathcal{A}$  is formal if  $\mathcal{A}$  is free. So it is natural to ask the following.

**Problem 2** If  $D^p(\mathcal{A})$  is free for some  $p$ ,  $3 \leq p \leq \ell - 2$ , then is  $\mathcal{A}$  formal?

By the same reason above, this problem becomes a problem when  $\ell = 6$  and  $D^3(\mathcal{A})$  is free.

Also, it is natural to ask to which extent we can show the results for free arrangements to the freeness of  $D^p(\mathcal{A})$ . I proved some of them. The first is Saito's criterion.

**Theorem 2** (Saito's criterion, [1]). Let  $s_p^\ell := {}_\ell C_p$ . Let  $\theta_1, \dots, \theta_{s_p^\ell} \in D^p(\mathcal{A}, \mathfrak{m})$ . Let  $M$  be a coefficient matrix of  $\theta_1, \dots, \theta_{s_p^\ell}$ . Then

$$\det M \in \text{SQ}(\mathcal{A}, \mathfrak{m})^{s_{p-1}^{\ell-1}}.$$

Moreover, the following three conditions are equivalent:

- (1)  $\theta_1, \dots, \theta_{s_p^\ell}$  form a basis for  $D^p(\mathcal{A}, \mathfrak{m})$ .
- (2)  $\det M = cQ(\mathcal{A}, \mathfrak{m})^{s_{p-1}^{\ell-1}}$  for some non-zero  $c \in \mathbb{K}$ .
- (3) (When all  $\theta_i$ 's are homogeneous)  $\theta_1, \dots, \theta_{s_p^\ell}$  are  $S$ -independent, and  $\sum_{i=1}^{s_p^\ell} \det \theta_i = s_{p-1}^{\ell-1} |\mathfrak{m}|$ .

Let  $s_p^\ell := {}_\ell C_p$ ,  $B$  be Terao's polynomial  $B$  with respect to  $(\mathcal{A}, H)$ , and  $t_p^\ell := s_p^\ell - s_{p-1}^{\ell-1} = s_{p-1}^{\ell-1}$ . We say that  $\mathcal{A}$  is  $p$ -free if  $D^p(\mathcal{A})$  is free. If  $\mathcal{A}$  is  $p$ -free with homogeneous basis  $\theta_1, \dots, \theta_{s_p^\ell}$ , then  $\exp^p(\mathcal{A}) := (\deg \theta_1, \dots, \deg \theta_{s_p^\ell})$ . Then we can show Terao's addition-deletion-restriction theorems.

**Theorem 3** (Addition-theorem, [1]). Let  $d := |\mathcal{A}| - 1 - |\mathcal{A}^H|$ . Assume that  $\mathcal{A}' := \mathcal{A} \setminus \{H\}$  is  $p$ -free,  $\mathcal{A}^H$  is both  $p$  and  $(p-1)$ -free,  $\exp^p(\mathcal{A}') = (\exp^{p-1}(\mathcal{A}^H) + (d)^{s_{p-1}^{\ell-1}}) \sqcup \exp^p(\mathcal{A}^H)$ . Then  $\mathcal{A}$  is  $p$ -free with

$$\exp^p(\mathcal{A}) = (\exp^{p-1}(\mathcal{A}^H) + (d+1)^{s_{p-1}^{\ell-1}}) \sqcup \exp^p(\mathcal{A}^H).$$

Moreover, there is a basis  $\theta_1, \dots, \theta_{s_p^{\ell-1}}, \varphi_1, \dots, \varphi_{s_{p-1}^{\ell-1}}$  for  $D^p(\mathcal{A}')$  such that

$$\theta_1, \dots, \theta_{s_p^{\ell-1}}, \alpha_H \varphi_1, \dots, \alpha_H \varphi_{s_{p-1}^{\ell-1}}$$

form a basis for  $D^p(\mathcal{A})$ ,

$$\rho^H(\theta_1), \dots, \rho^H(\theta_{s_p^{\ell-1}})$$

form a basis for  $D^p(\mathcal{A}^H)$ , and

$$\overline{\partial^p}(\varphi_1), \dots, \overline{\partial^p}(\varphi_{s_{p-1}^{\ell-1}})$$

form a basis for  $D^{p-1}(\mathcal{A}^H)\overline{B}$ .

**Theorem 4** (Deletion theorem, [1]). *Let  $d := |\mathcal{A}'| - |\mathcal{A}^H|$ . Assume that  $\mathcal{A}$  is  $p$ -free,  $\mathcal{A}^H$  is both  $p$  and  $(p-1)$ -free,  $\exp^p(\mathcal{A}) = (\exp^{p-1}(\mathcal{A}^H) + (d+1)s_{p-1}^{\ell-1}) \sqcup \exp^p(\mathcal{A}^H)$ . Then  $\mathcal{A}'$  is  $p$ -free with*

$$\exp^p(\mathcal{A}') = (\exp^{p-1}(\mathcal{A}^H) + (d)s_{p-1}^{\ell-1}) \sqcup \exp^p(\mathcal{A}^H).$$

Moreover, there is a basis

$$\theta_1, \dots, \theta_{s_p^{\ell-1}}, \alpha_H \varphi_1, \dots, \alpha_H \varphi_{s_{p-1}^{\ell-1}}$$

for  $D^p(\mathcal{A})$  such that

$$\theta_1, \dots, \theta_{s_p^{\ell-1}}, \varphi_1, \dots, \varphi_{s_{p-1}^{\ell-1}}$$

form a basis for  $D^p(\mathcal{A}')$ ,

$$\rho^H(\theta_1), \dots, \rho^H(\theta_{s_p^{\ell-1}})$$

form a basis for  $D^p(\mathcal{A}^H)$ , and

$$\overline{\partial^p}(\varphi_1), \dots, \overline{\partial^p}(\varphi_{s_{p-1}^{\ell-1}})$$

form a basis for  $D^{p-1}(\mathcal{A}^H)\overline{B}$ .

**Theorem 5** (Restriction theorem, [1]). *Assume that  $\mathcal{A}$  and  $\mathcal{A}'$  are both  $p$ -free. Then  $\mathcal{A}^H$  is  $p$  and  $(p-1)$ -free. Explicitly, there is a basis*

$$\theta_1, \dots, \theta_{s_p^{\ell-1}}, \varphi_1, \dots, \varphi_{s_{p-1}^{\ell-1}}$$

for  $D^p(\mathcal{A}')$  such that

$$\theta_1, \dots, \theta_{s_p^{\ell-1}}, \alpha_H \varphi_1, \dots, \alpha_H \varphi_{s_{p-1}^{\ell-1}}$$

form a basis for  $D^p(\mathcal{A})$ ,

$$\overline{\theta}_1, \dots, \overline{\theta}_{s_p^{\ell-1}}$$

form a basis for  $D^p(\mathcal{A}^H)$ , and

$$\overline{\partial^p}(\varphi_1)/\overline{B}, \dots, \overline{\partial^p}(\varphi_{s_{p-1}^{\ell-1}})/\overline{B}$$

for  $D^{p-1}(\mathcal{A}^H)$ .

Also we can show Yoshinaga's criterion and the division theorems.

**Theorem 6** (Yoshinaga-type criterion for  $p$ -freeness, [1]).  *$D_H^p(\mathcal{A})$  is free if and only if there is  $H \in \mathcal{A}$  such that*

- (1)  $(\mathcal{A}^H, \mathfrak{m}^H)$  is  $p$ -free, and
- (2) the upper  $b_2$ -equality  $b_2^0(\mathcal{A}) = b_2(\mathcal{A}^H, \mathfrak{m}^H)$  holds.

**Theorem 7** (Division theorem for  $p$ -freeness).  *$\mathcal{A}$  is  $p$ -free if  $\mathcal{A}^H$  is  $p$  and  $(p-1)$ -free, and the  $b_2$ -equality holds for  $(\mathcal{A}, H)$ .*

However, we do not know whether the above formulations of several results for usual free arrangements are the right one. It could be an easier way to formulate them. So we can ask the following.

**Problem 3.** Give  $p$ -freeness version of the other results for free arrangements. Also, find a better formulation of the results above.

Also, we have to note that, if someone shows that the freeness of  $\mathcal{A}$  is the same as  $p$ -freeness for  $3 \leq p \leq \ell - 3$ , then the above follows immediately from the usual results for freeness. So the most important problem is the (non-)equivalent of freeness and  $p$ -freeness.

**References**

- [1] T. Abe, Fundamentals of higher order logarithmic derivation modules of hyperplane arrangements. In preparation.

## 2: Classification of free multiplicities on braid arrangements

(suggested by Michael DiPasquale)

We propose the problem of classifying free multiplicities on braid arrangements. We explain the progress that has been made so far on this problem and a conjecture for the complete classification, which we made in [7].

Let  $V \cong \mathbb{K}^{\ell+1}$  be a vector space over a field  $\mathbb{K}$  of characteristic zero,  $V^*$  its dual space and  $S = \text{Sym}(V^*) \cong \mathbb{K}[x_0, \dots, x_\ell]$ . The *braid arrangement of type*  $A_\ell \subset V$  is defined as  $A_\ell = \cup_{0 \leq i < j \leq \ell} H_{ij}$ , where  $H_{ij} = \ker(x_i - x_j)$ . A *multiplicity* on  $A_\ell$  is a map  $m : \{H_{ij}\} \rightarrow \mathbb{Z}_{>0}$ ; we will set  $m_{ij} = m(H_{ij})$ . The pair  $(A_\ell, m)$  is called a multi-arrangement. It is standard to represent the braid multi-arrangement  $(A_\ell, m)$  as a complete graph on  $\ell + 1$  vertices labeled  $\{0, \dots, \ell\}$  with the edge  $\{i, j\}$  labeled by the multiplicity  $m_{ij}$ .

To the multi-arrangement  $(A_\ell, m)$  we associate its *module of multi-derivations*. The module of derivations on  $S$  is defined by  $\text{Der}_{\mathbb{K}}(S) = \bigoplus_{i=1}^{\ell} S \partial_{x_i}$ ; this is a free  $S$ -module with an action on  $S$  by partial differentiation. The module of multi-derivations of  $(A, m)$  is

$$D(A, m) := \{\theta \in \text{Der}_{\mathbb{K}}(S) : \theta(\alpha_H) \in \langle \alpha_H^{m(H)} \rangle \text{ for all } H \in A\},$$

where  $\langle \alpha_H^{m(H)} \rangle \subset S$  is the ideal generated by  $\alpha_H^{m(H)}$ . If  $D(A, m)$  is a free  $S$ -module, then we say  $(A, m)$  is free or  $m$  is a free multiplicity of  $A$ .

Free multiplicities on braid arrangements have been studied since the introduction of the module of logarithmic differentials by Saito [9], due to their importance in the theory of Coxeter arrangements. Via Yoshinaga's celebrated freeness criterion [14], free multiplicities on Coxeter arrangements are integral to studying freeness of their *deformations* (first studied by Stanley [11], see also Athanasiadis [4]). Terao made a major breakthrough in [12], showing that the constant multiplicity on any Coxeter arrangement is free and determining the corresponding exponents. Subsequently, many authors studied freeness of 'almost-constant' multiplicities on Coxeter and braid arrangements [10, 12, 13, 1, 3]. In the setting of the braid arrangement, Abe-Nuida-Numata [2] characterized a general class of free multiplicities which we shall call *ANN multiplicities*. Given non-negative integers  $n_0, \dots, n_\ell$  and integers  $\epsilon_{ij} \in \{-1, 0, 1\}$  for all  $0 \leq i < j \leq \ell$ , an ANN multiplicity is a multiplicity satisfying

1.  $m_{ij} = n_i + n_j + \epsilon_{ij}$  and
2.  $m_{ij} \leq m_{ik} + m_{jk} + 1$  for every triple  $i, j, k$ .

In [2], Abe-Nuida-Numata prove that ANN multiplicities are free if and only if the graph with edges signed according to  $\epsilon_{ij}$  is *signed-eliminable*. In [7] it was shown that free ANN multiplicities account for all free multiplicities in a large region.

**Theorem 1.** [7] *A multiplicity  $m$  is in the balanced cone if it satisfies the inequalities in 2; namely  $m_{ij} \leq m_{ik} + m_{jk} + 1$  for every triple  $i, j, k$ . Suppose  $m$  is a multiplicity on  $A_\ell$  in the balanced cone. Then  $m$  is free a free multiplicity on  $A_\ell$  if and only if  $m$  is a free ANN multiplicity.*

We say that  $i$  is a *free vertex* of  $m$  if  $m_{ij} + m_{ik} \leq m_{jk} + 1$  for all pairs  $j, k \neq i$  with  $0 \leq j < k \leq \ell$ . Put  $\mathcal{A}'_{\ell-1} = \bigcup_{\substack{0 \leq j < k \leq \ell \\ j, k \neq i}} H_{jk}$  and  $m'$  the restriction of  $m$  to the hyperplanes

of  $\mathcal{A}'_{\ell-1}$ . If  $i$  is a free vertex then  $(\mathcal{A}_\ell, m)$  is free if and only if  $(\mathcal{A}'_{\ell-1}, m')$  is free (c.f. [7, Theorem 6.2]). In [6] we show the following classification of free multiplicities on  $\mathcal{A}_3$ .

**Theorem 2.** [6] *Suppose  $m$  is a free multiplicity on  $\mathcal{A}_3$ . Then  $m$  is either a free ANN multiplicity or  $m$  has a free vertex.*

In [7] we conjecture that Theorem 2 generalizes to arbitrary braid arrangements.

**Conjecture 3.** [7] *Suppose  $m$  is a free multiplicity on  $\mathcal{A}_\ell$ . Then either  $m$  is a free ANN multiplicity or  $m$  has a free vertex. By Theorem 1, it suffices to show that if  $m$  is a free multiplicity that is not in the balanced cone, then it necessarily has a free vertex.*

If Conjecture 3 is true, then it completes the classification of free multiplicities on braid arrangements (details may be found in [7]).

Let  $J(\mathcal{A}_\ell, m) = \langle (\chi_i - \chi_j)^{m_{ij}} : 0 \leq i < j \leq \ell \rangle$ . In [6], it is proved that, if  $D(\mathcal{A}_3, m)$  is free, then it is a third syzygy module of the quotient  $S/J(\mathcal{A}_\ell, m)$ . In fact, if  $D(\mathcal{A}_3, m)$  is free, then a free resolution of  $S/J(\mathcal{A}_3, m)$  can be explicitly constructed from  $D(\mathcal{A}_3, m)$  and its localizations at sub-braid arrangements of  $\mathcal{A}_3$ .

This gives a strong obstruction to freeness. In [6], we proved that if  $m$  is a free multiplicity on  $\mathcal{A}_3$  which is not in the balanced cone then a free resolution of  $S/J(\mathcal{A}_3, m)$  could only be constructed from the free module  $D(\mathcal{A}_3, m)$  if  $m$  has a free vertex (otherwise certain Koszul syzygies on the generators of  $J(\mathcal{A}_3, m)$  are obstructions).

A generalization turns out to be true in higher dimensions (this is still unpublished work): that is, if  $D(\mathcal{A}_\ell, m)$  is free then a free resolution of  $S/J(\mathcal{A}_\ell, m)$  can be explicitly constructed from  $D(\mathcal{A}_\ell, m)$  and its localizations at sub-braid arrangements. However, it is not yet clear how one could unravel the combinatorics of this resolution to prove that if  $m$  is not in the balanced cone then it must have a free vertex.

There is one more piece of information that could possibly come into play here -  $S/J(\mathcal{A}_\ell, m)$  is an ideal generated by powers of linear forms. As such, its Hilbert function is related to that of a fat point configuration in  $\mathbb{P}^{\ell-1}$  by Emsalem and Iarrobino [8]. Hilbert functions of fat points are highly non-trivial, but the fat point configurations relevant to this problem have a great deal of structure. It may be that techniques such as those developed by Cooper-Harbourne-Tietler [5] would allow the computation of the Hilbert function of  $S/J(\mathcal{A}_\ell, m)$ .

One could then proceed by induction and contradiction: assume that  $m$  is in the balanced cone but has no free vertex. Then construct a free resolution of  $S/J(\mathcal{A}_\ell, m)$  from  $D(\mathcal{A}, m)$  and its localizations at sub-braid arrangements. By induction, the exponents of the localized multi-arrangements are known. Thus, if one knows the Hilbert function of  $S/J(\mathcal{A}_\ell, m)$  then the Hilbert function of  $D(\mathcal{A}, m)$  can be deduced. Show that this cannot be the Hilbert function of a free module (for example, show that the numerator of the Hilbert series has negative coefficients). If this approach is successful, it will require a great deal of combinatorial book-keeping.

**Question 1:** Is the approach outlined above a viable approach to prove Conjecture 3? If so, are there effective ways to encode the combinatorics of the free resolution constructed from  $D(\mathcal{A}_\ell, m)$  and the Hilbert function of  $S/J(\mathcal{A}_\ell, m)$ ? Are there other approaches that are simpler?

## References

- [1] Takuro Abe and Masahiko Yoshinaga. Coxeter multiarrangements with quasi-constant multiplicities. *J. Algebra*, 322(8):2839–2847, 2009. ISSN 0021-8693,1090-266X. doi: 10.1016/j.jalgebra.2009.07.025. URL <https://doi-org.nmsu.idm.oclc.org/10.1016/j.jalgebra.2009.07.025>.
- [2] Takuro Abe, Koji Nuida, and Yasuhide Numata. Signed-eliminable graphs and free multiplicities on the braid arrangement. *J. Lond. Math. Soc. (2)*, 80(1):121–134, 2009. ISSN 0024-6107,1469-7750. doi: 10.1112/jlms/jdp019. URL <https://doi-org.nmsu.idm.oclc.org/10.1112/jlms/jdp019>.
- [3] Takuro Abe, Hiroaki Terao, and Atsushi Wakamiko. Equivariant multiplicities of Coxeter arrangements and invariant bases. *Adv. Math.*, 230(4-6):2364–2377, 2012. ISSN 0001-8708,1090-2082. doi: 10.1016/j.aim.2012.04.015. URL <https://doi-org.nmsu.idm.oclc.org/10.1016/j.aim.2012.04.015>.
- [4] Christos A. Athanasiadis. Deformations of Coxeter hyperplane arrangements and their characteristic polynomials. In *Arrangements—Tokyo 1998*, volume 27 of *Adv. Stud. Pure Math.*, pages 1–26. Kinokuniya, Tokyo, 2000.
- [5] Susan Cooper, Brian Harbourne, and Zach Teitler. Combinatorial bounds on Hilbert functions of fat points in projective space. *J. Pure Appl. Algebra*, 215(9):2165–2179, 2011. ISSN 0022-4049,1873-1376. doi: 10.1016/j.jpaa.2010.12.006. URL <https://doi-org.nmsu.idm.oclc.org/10.1016/j.jpaa.2010.12.006>.
- [6] Michael DiPasquale, Christopher A. Francisco, Jeffrey Mermin, and Jay Schweig. Free and non-free multiplicities on the  $A_3$  arrangement. *J. Algebra*, 544:498–532, 2020. ISSN 0021-8693,1090-266X. doi: 10.1016/j.jalgebra.2019.10.029. URL <https://doi-org.nmsu.idm.oclc.org/10.1016/j.jalgebra.2019.10.029>.
- [7] Michael Robert DiPasquale. Inequalities for free multi-braid arrangements. *Proc. Japan Acad. Ser. A Math. Sci.*, 94(4):36–41, 2018. ISSN 0386-2194. doi: 10.3792/pjaa.94.36. URL <https://doi-org.nmsu.idm.oclc.org/10.3792/pjaa.94.36>.
- [8] J. Emsalem and A. Iarrobino. Inverse system of a symbolic power. I. *J. Algebra*, 174(3):1080–1090, 1995. ISSN 0021-8693,1090-266X. doi: 10.1006/jabr.1995.1168. URL <https://doi-org.nmsu.idm.oclc.org/10.1006/jabr.1995.1168>.
- [9] Kyoji Saito. On the uniformization of complements of discriminant loci. In *Conference Notes. Amer. Math. Soc. Summer Institute, Williamstown*, 1975.
- [10] Louis Solomon and Hiroaki Terao. The double Coxeter arrangement. *Comment. Math. Helv.*, 73(2):237–258, 1998. ISSN 0010-2571. doi: 10.1007/s000140050054. URL <http://dx.doi.org/10.1007/s000140050054>.
- [11] Richard P. Stanley. Hyperplane arrangements, interval orders, and trees. *Proc. Nat. Acad. Sci. U.S.A.*, 93(6):2620–2625, 1996. ISSN 0027-8424. doi: 10.1073/pnas.93.6.2620. URL <https://doi-org.nmsu.idm.oclc.org/10.1073/pnas.93.6.2620>.
- [12] Hiroaki Terao. Multiderivations of Coxeter arrangements. *Invent. Math.*, 148(3):659–674, 2002. ISSN 0020-9910,1432-1297. doi: 10.1007/s002220100209. URL <https://doi-org.nmsu.idm.oclc.org/10.1007/s002220100209>.

- [13] Masahiko Yoshinaga. The primitive derivation and freeness of multi-Coxeter arrangements. *Proc. Japan Acad. Ser. A Math. Sci.*, 78(7):116–119, 2002. ISSN 0386-2194. URL <http://projecteuclid.org.nmsu.idm.oclc.org/euclid.pja/1148392632>.
- [14] Masahiko Yoshinaga. Characterization of a free arrangement and conjecture of Edelman and Reiner. *Invent. Math.*, 157(2):449–454, 2004. ISSN 0020-9910,1432-1297. doi: 10.1007/s00222-004-0359-2. URL <https://doi-org.nmsu.idm.oclc.org/10.1007/s00222-004-0359-2>.

### 3: A systematic study of matroid orientations using SAT solvers

(suggested by Lukas Kühne)

The topology of the complexified complement of a real hyperplane arrangement is a central object of study in the combinatorics and topology of arrangements. Many invariants—such as the cohomology ring of the complement, encoded in the Orlik–Solomon algebra—are determined by the underlying *matroid* alone. However, finer topological invariants, most notably the Betti numbers of the *Milnor fiber*, are believed to carry strictly more information. We propose a systematic computational investigation: use modern SAT solvers to enumerate, for each matroid in a small-cases database, all non-isomorphic orientations (i.e., oriented matroids), and then study the resulting invariants of the Salvetti complex and the Milnor fiber.

**Matroids and oriented matroids.** A matroid  $M = (E, \mathcal{B})$  on a finite ground set  $E$  abstracts linear independence. Every real hyperplane arrangement  $\mathcal{A}$  in  $\mathbb{R}^\ell$  determines a matroid  $M(\mathcal{A})$  via its intersection lattice. A *chirotope* of rank  $r$  on  $E$  is an alternating map  $\chi: E^r \rightarrow \{+, -, 0\}$  satisfying the Grassmann–Plücker exchange relations [1]. A chirotope encodes the signs of the maximal minors of the matrix of normal vectors of  $\mathcal{A}$  and captures the signed combinatorial structure of the arrangement. Two chirotopes are *reorientation equivalent* if one is obtained from the other by flipping the signs of a subset  $A \subseteq E$ . An *oriented matroid*  $\mathcal{M}$  is a reorientation class of chirotopes together with the underlying matroid  $M$ .

A fundamental—and computationally hard—problem is that a single matroid may admit many non-isomorphic orientations:

**Question 1:** For each matroid  $M$  in the database of matroids on at most  $n$  elements and rank at most  $r$ , determine all oriented matroids  $\mathcal{M}$  with underlying matroid  $M$ , up to reorientation and automorphism. How does the number of non-isomorphic orientations depend on the combinatorial structure of  $M$ ? Are there matroids that admit no orientation (in particular are non-realizable over  $\mathbb{R}$ ), and can SAT solvers detect and certify non-orientability efficiently?

**SAT-based enumeration.** The chirotope axioms—Grassmann–Plücker relations and the non-degeneracy conditions encoding which  $r$ -subsets are bases—translate naturally into a Boolean satisfiability (SAT) problem. One introduces a Boolean variable for each sign  $\chi(e_{i_1}, \dots, e_{i_r}) \in \{+, -\}$  over the bases of  $M$ , and encodes each exchange relation as a clause. Modern SAT solvers (CaDiCaL, Kissat, etc.) together with symmetry-breaking techniques can enumerate all solutions modulo the reorientation group and the automorphism group of  $M$ .

Comprehensive databases of matroids are available for small parameters: all matroids on up to 8 elements are known, and the rank-3 case has been studied up to 12 or more points. A systematic SAT-based sweep over such a database would yield a complete census of oriented matroids in these ranges, far exceeding the partial classifications currently in the literature.

**The Salvetti complex.** To each oriented matroid  $\mathcal{M}$  (in particular to each real arrangement  $\mathcal{A}$ ) one associates the *Salvetti complex*  $\text{Sal}(\mathcal{M})$ , a finite CW-complex introduced by Salvetti [5] that is a deformation retract of the complexified complement  $M(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H \otimes \mathbb{C}$ . Its cells are indexed by pairs  $(F, C)$  where  $F$  is a face of the real arrangement and  $C$  is

a chamber. The complex  $\text{Sal}(\mathcal{M})$  depends only on the oriented matroid  $\mathcal{M}$ , so different orientations of the same matroid yield potentially non-isomorphic Salvetti complexes and distinct twisted cellular chain complexes.

The Milnor fiber  $F(\mathcal{A}) = Q^{-1}(1)$  of the arrangement, where  $Q = \prod_{H \in \mathcal{A}} \alpha_H$ , is a finite cyclic cover of  $M(\mathcal{A}_{\mathbb{C}})$ . Its homology can be computed from the cellular chain complex of  $\text{Sal}(\mathcal{M})$  twisted by a character  $\omega: \pi_1(M(\mathcal{A}_{\mathbb{C}})) \rightarrow \mathbb{Z}/n\mathbb{Z}$ , where  $n = |\mathcal{A}|$  (see [4] and [3]). This twisted chain complex depends on the oriented matroid. Remarkably, Mücke and Yoshinaga [2] recently showed that the homotopy type of the Milnor fiber is in fact entirely determined by the oriented matroid  $\mathcal{M}$ , via a poset quasi-fibration whose domain subdivides the Salvetti complex. Moreover, their construction extends beyond realizable oriented matroids, yielding a purely combinatorial notion of the Milnor fiber for any oriented matroid. This means that every oriented matroid produced by the SAT enumeration carries a well-defined Milnor fiber, whether or not it arises from an actual hyperplane arrangement.

**Question 2:** For each oriented matroid  $\mathcal{M}$  produced by the SAT enumeration, compute the Betti numbers  $b_k(F(\mathcal{A}); \mathbb{Z})$  of the corresponding Milnor fiber using the construction of Mücke–Yoshinaga. Which cases are computationally accessible, and what structural features of  $\mathcal{M}$  (e.g., supersolvability, local freeness) make the computation tractable?

**Combinatorial determination of Milnor fiber Betti numbers.** While the Betti numbers of  $M(\mathcal{A}_{\mathbb{C}})$  are determined by the matroid  $M(\mathcal{A})$  alone—they are the coefficients of the Poincaré polynomial—the individual Betti numbers  $b_k(F(\mathcal{A}); \mathbb{Z})$  are the subject of a long-standing open problem:

**Conjecture 1.** *The Betti numbers  $b_k(F(\mathcal{A}); \mathbb{Z})$  are determined by the matroid  $M(\mathcal{A})$  alone.*

Evidence for this conjecture comes from large families (generic arrangements, free arrangements, supersolvable arrangements), but no proof or counterexample is known in general. Papadima and Suciu [4] expressed the Betti numbers of  $F$  via resonance and monodromy data of the Aomoto complex, a construction that in principle depends on the oriented matroid. A systematic computation for all oriented matroids arising from small matroids would either produce a counterexample—two orientations of the same matroid giving different Milnor fiber Betti numbers—or provide strong computational evidence for the conjecture:

**Question 3:** Among all matroids  $M$  admitting multiple non-isomorphic orientations (as found in Question 1), do the corresponding Milnor fibers always have the same Betti numbers? If not, exhibit an explicit matroid  $M$  and two orientations  $\mathcal{M}_1, \mathcal{M}_2$  with  $b_k(F(\mathcal{A}_1)) \neq b_k(F(\mathcal{A}_2))$  for some  $k$ , thereby disproving the conjecture.

A positive answer to Question 3—even a single matroid with two orientations yielding different Milnor fiber Betti numbers—would be a major result. Conversely, finding agreement in all small cases would provide compelling evidence for the conjecture and sharpen intuition about which combinatorial features of the oriented matroid control the Milnor fiber topology.

## References

- [1] BJÖRNER, A., LAS VERGNAS, M., STURMFELS, B., WHITE, N., AND ZIEGLER, G. M. *Oriented Matroids*, 2nd ed. Encyclopedia of Mathematics and its Applications, vol. 46. Cambridge University Press, 1999.
- [2] MÜCKSCH, P., AND YOSHINAGA, M. Milnor fibrations and oriented matroids. Preprint, arXiv:2508.15331 (2025).
- [3] COHEN, D. C., AND ORLIK, P. Arrangements and local systems. *Math. Res. Lett.* 7 (2000), 299–316.
- [4] PAPADIMA, S., AND SUCIU, A. I. The Milnor fibration of a hyperplane arrangement: from modular resonance to algebraic monodromy. *Proc. Lond. Math. Soc.* 114 (2017), 961–1004.
- [5] SALVETTI, M. Topology of the complement of a real arrangement in  $\mathbb{C}^N$ . *Trans. Amer. Math. Soc.* 388 (1993), 167–183.

## 4: Projective dimension and Zariski-open subsets of the realization space

(suggested by Paul Mücksch)

Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be a hyperplane arrangement in the  $\mathbb{K}$ -vector space  $V = \mathbb{K}^\ell$ , defined by linear forms  $\alpha_i \in V^*$  ( $1 \leq i \leq n$ ). Let  $S = \mathbb{K}[x_1, \dots, x_\ell]$  be the coordinate ring of  $V$ , and let  $D(\mathcal{A}) = \{\theta \in \text{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in \alpha_H S\}$  denote the module of  $\mathcal{A}$ -derivations. Terao's conjecture (see, e.g., [6, Conj. 4.138]) asserts the following: if  $\mathcal{B} = \{H'_1, \dots, H'_n\}$  is another arrangement in  $V$  whose intersection lattice  $L(\mathcal{B}) = \{\cap_{j \in T} H'_j \mid T \subseteq [n]\}$  is isomorphic to  $L(\mathcal{A})$  as a poset, then  $D(\mathcal{A})$  is free over  $S$  if and only if  $D(\mathcal{B})$  is. This conjecture remains open.

Positive evidence is provided by Terao's factorization theorem [8], which asserts that if the graded module  $D(\mathcal{A})$  is free and generated in degrees  $(e_1, \dots, e_\ell)$ , then the characteristic polynomial, an invariant of the poset  $L(\mathcal{A})$ , factors over  $\mathbb{Z}$  as  $\chi(\mathcal{A}, t) = (t - e_1) \cdots (t - e_\ell)$ . Moreover, there are many subclasses of free arrangements, such as inductively free [7], divisionally free [1] and additively free arrangements [2] for which the conjecture is true.

In contrast, if one drops the freeness assumption on the derivation module, the sequence of degrees of a minimal set of generators is not combinatorial. An example was first given by Ziegler [12, Ex. 8.7]. For the pair of arrangements

$$\begin{aligned} \mathcal{Z}_1 = \{ & \{x = 0\}, \{y = 0\}, \{z = 0\}, \{x + y + z = 0\}, \\ & \{2x + y + z = 0\}, \{2x + 3y + z = 0\}, \\ & \{2x + 3y + 4z = 0\}, \{3x + 5z = 0\}, \{3x + 4y + 5z = 0\} \} \end{aligned}$$

and

$$\begin{aligned} \mathcal{Z}_2 = \{ & \{x = 0\}, \{y = 0\}, \{z = 0\}, \{x + y + z = 0\}, \\ & \{2x + y + z = 0\}, \{2x + 3y + z = 0\}, \\ & \{2x + 3y + 4z = 0\}, \{x + 3z = 0\}, \{x + 2y + 3z = 0\} \}, \end{aligned}$$

one has  $L(\mathcal{Z}_1) \cong L(\mathcal{Z}_2)$  as posets, but  $D(\mathcal{Z}_1)$  is minimally generated in degrees  $(1, 6, 6, 6, 6, 6, 6)$ , whereas  $D(\mathcal{Z}_2)$  is minimally generated in degrees  $(1, 5, 6, 6, 6)$ . However, the coarser algebraic invariant given by the projective dimension of the derivation module still coincides.

Now let  $L = L(\mathcal{A})$ . Then the space

$$\mathcal{R}(L) = \{\mathcal{B} \text{ in } V \mid L(\mathcal{B}) = L\}$$

of all arrangements with intersection lattice  $L$  is called the realization space of  $L$ . It has the structure of a quasi-projective variety. Yuzvinsky [11] showed that the subset of free arrangements in  $\mathcal{R}(L)$  is Zariski-open by considering the sheaf on  $L$  given by  $\mathcal{D}: L(\mathcal{A}) \rightarrow \text{Mod}_S, (X \subseteq Y) \mapsto (D(\mathcal{A}_X) \hookrightarrow D(\mathcal{A}_Y))$ , where  $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$ , and by characterizing the freeness of a locally free arrangement in terms of the vanishing of the cohomology groups  $H^i(\mathcal{D})$  in the middle degrees  $1 \leq i \leq \ell - 2$ .

In [5], the sheaf  $\mathcal{D}$  was studied further, and an alternative proof of Yuzvinsky's freeness characterization was given. Moreover, this characterization generalizes to one for the projective dimension of  $D(\mathcal{A})$ .

**Theorem 1** ([5, Thm. 1.5]). *The following conditions are equivalent:*

- (i)  $\text{pd}(\mathcal{D}) \leq p$ ;
- (ii)  $H^n(\mathcal{D}) = 0$  for  $0 < n < \ell - 1 - p$ .

This leads to the following variant of Terao’s conjecture, asking whether even the projective dimension of the derivation module is combinatorial (see also [3, Prob. 6.3(3)]).

**Question 1:** Let  $\mathcal{A}, \mathcal{B}$  be arrangements in  $V$  with  $L(\mathcal{A}) \cong L(\mathcal{B})$ , and denote their corresponding sheaves of logarithmic vector fields on  $L$  by  $\mathcal{D}_{\mathcal{A}}$  and  $\mathcal{D}_{\mathcal{B}}$ , respectively. Does

$$H^i(\mathcal{D}_{\mathcal{A}}) = 0 \iff H^i(\mathcal{D}_{\mathcal{B}}) = 0$$

hold for all  $i$ ?

A combinatorial characterization of projective dimension 1 for graphic arrangements was recently established in [4].

Finally, in view of Yuzvinsky’s result that free arrangements form a Zariski-open subset of  $\mathcal{R}(L)$ , one may ask the following.

**Question 2:** Do arrangements with projective dimension at most  $p$  form a Zariski-open subset of  $\mathcal{R}(L)$ ?

## References

- [1] ABE, T. Divisionally free arrangements of hyperplanes. *Invent. Math.* 204, 1 (2016), 317–346.
- [2] ABE, T. Addition-deletion theorem for free hyperplane arrangements and combinatorics. *J. Algebra* 610 (2022), 1–17.
- [3] ABE, T. Projective dimensions of hyperplane arrangements. *Trans. Am. Math. Soc.* 377, 11 (2024), 7793–7827.
- [4] ABE, T., KÜHNE, L., MÜCKSCH, P., AND MÜHLHERR, L. Projective dimension of weakly chordal graphic arrangements, 2025.
- [5] MÜCKSCH, P. On Yuzvinsky’s lattice sheaf cohomology for hyperplane arrangements. *Mathematische Annalen* (2022), <https://doi.org/10.1007/s00208--022--02499--1>.
- [6] ORLIK, P., AND TERAQ, H. *Arrangements of Hyperplanes*. Grundlehren der mathematischen Wissenschaften. Springer, 1992.
- [7] TERAQ, H. Arrangements of hyperplanes and their freeness I. *J. Fac. Sci. Univ. Tokyo* 27 (1980), 293–320.
- [8] TERAQ, H. Generalized exponents of a free arrangement of hyperplanes and Shepherd-Todd-Brieskorn formula. *Invent. Math.* 63, 1 (1981), 159–179.

- [9] YUZVINSKY, S. Cohomology of local sheaves on arrangement lattices. *Proc. Amer. Math. Soc.* 112, 4 (1991), 1207–1217.
- [10] YUZVINSKY, S. The first two obstructions to the freeness of arrangements. *Trans. Amer. Math. Soc.* 335, 1 (1993), 231–244.
- [11] YUZVINSKY, S. Free and locally free arrangements with a given intersection lattice. *Proc. Amer. Math. Soc.* 118, 3 (1993), 745–752.
- [12] ZIEGLER, G. M. Combinatorial construction of logarithmic differential forms. *Adv. Math.* 76, 1 (1989), 116–154.

## 5: Problems of Buildingset arrangements

(suggested by Leonie Mühlherr)

In [1], Cuntz and Kühne defined the following:

**Definition 1.** Let  $G = (N, E)$  be a simple hypergraph with set of vertices  $N = [n] = \{1, 2, \dots, n\}$  and set of edges  $E \subset 2^N$ . We define the **connected subgraph arrangement**  $\mathcal{A}_G$  in  $V = \mathbb{Q}^n$  as

$$\mathcal{A}_G := \{H_I \mid \emptyset \neq I, G[I] \text{ is connected}\},$$

where  $H_I$  is the hyperplane  $H_I = \ker \sum_{i \in I} x_i$ .

From this definition, interesting arrangements can be recovered as connected subgraph arrangements for special graphs such as the complete or the path graph. Cuntz and Kühne analysed different properties of these arrangements, notably the property of factoredness.

**Definition 2** ([3], Definition 2.66). Let  $\pi = (\pi_1, \dots, \pi_s)$  be a partition of  $\mathcal{A}$ . The partition  $\pi$  is called **independent** if for any choice  $H_i \in \pi_i$  for  $1 \leq i \leq s$ , the resulting  $s$  hyperplanes are linearly independent.

Let  $X \in L(\mathcal{A})$ . The **induced partition**  $\pi_X$  of  $\mathcal{A}_X$  is given by the nonempty blocks of the form  $\pi_i \cap \mathcal{A}_X$ . The partition  $\pi$  is a **factorization** of  $\mathcal{A}$  if

1.  $\pi$  is independent
2. For each  $X \in L(\mathcal{A}) \setminus V$ , the induced partition  $\pi_X$  admits a block which is a singleton.

We call  $\mathcal{A}$  **factored** if it admits a factorization.

For the connected subgraph arrangements, there is a characterization of factoredness:

**Theorem 3** ([1], Theorem 8.10). Let  $G$  be a connected graph. The connected subgraph arrangement  $\mathcal{A}_G$  is factored if and only if  $G$  is a path graph or a path-with-triangle graph  $\Delta_{n,1}$ .

See Figure 1 for an explanation of these graphs.



**Figure 1:** Examples of graphs  $P_6$  and  $\Delta_{6,1}$  corresponding to factored connected subgraph arrangements  $\Delta_{n,1}$

One can replace the graphs in Definition 1 by hypergraphs. Each of these arrangements is associated to a unique **Boolean building set** and each building set defines such an arrangements, thus, we call these arrangements **building set arrangements**.

**Definition 4** ([2], Lemma 3.9). *A family  $\mathcal{F}$  of subsets of  $[n]$  is a building set in the Boolean lattice  $2^{[n]}$  if and only if  $\mathcal{F}$  contains all singletons  $\{i\}$ ,  $i \in [n]$ , and the following condition holds: if  $F, F' \in \mathcal{F}$  and  $F \cap F' \neq \emptyset$ , then  $F \cup F' \in \mathcal{F}$ .*

Thus, we can ask:

**Question 1:** Is there a characterization of factored building set arrangements by hypergraph or building set properties?

A follow-up question concerns itself with algebraic implications of the factoredness of an arrangement.

**Question 2:** Is there a characterization of the numbers  $|\pi_i|$  (in the free case the exponents) by hypergraph or building set properties?

A less algebraic question that could be of interest is the following:

**Question:** The chambers of the braid arrangement are in 1-to-1 correspondence with permutations. Is there a combinatorial object (possibly related to buildingsets) that is enumerated by the chambers of the buildingset arrangements?

## References

- [1] CUNTZ, M., AND KÜHNE, L. On arrangements of hyperplanes from connected subgraphs, 2022.
- [2] FEICHTNER, E. M., AND STURMFELS, B. Matroid polytopes, nested sets and Bergman fans. *Port. Math. (N.S.)* 62, 4 (2005), 437–468.
- [3] ORLIK, P., AND TERAOKA, H. *Arrangements of hyperplanes*, vol. 300 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

## 6: Torsion in first homology group of the Milnor fiber boundary of arrangements

(suggested by Sakumi Sugawara)

Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be a central hyperplane arrangement in  $\mathbb{C}^3$  with the complement  $M(\mathcal{A}) = \mathbb{C}^3 \setminus \bigcup_{i=1}^n H_i$ . Let  $\alpha_i$  be a defining linear function of  $H_i$  and  $Q = \prod_{i=1}^n \alpha_i$ . This defines a homogeneous polynomial function  $Q : \mathbb{C}^3 \rightarrow \mathbb{C}$ . It is known that this restricts to the fiber bundle  $Q : M(\mathcal{A}) \rightarrow \mathbb{C}^*$ , which is called the Milnor fibration of the arrangement. The regular fiber  $F = Q^{-1}(1)$  is called the *Milnor fiber*, and its intersection with a sphere with large radius  $\partial F = Q^{-1}(1) \cap S_R^5$  is called the *Milnor fiber boundary*. The Milnor fiber boundary is an orientable closed 3-manifold.

The topology of the Milnor fiber boundary of a non-isolated surface singularity is studied in detail by Némethi-Szilárd in the book [1]. They also studied the Milnor fiber boundary of arrangements and obtained several results.

**Theorem 1.** (1) *The homeomorphism type of the Milnor fiber boundary is determined by its combinatorial structure.*

(2) *The first Betti number satisfies the following formula:*

$$b_1(\partial F) = \sum_{X \in L_2(\mathcal{A})} (1 + (|\mathcal{A}_X| - 2) \gcd(|\mathcal{A}_X|, n)),$$

where  $L_2(\mathcal{A})$  is the set of codimension two intersection of  $\mathcal{A}$  and  $\mathcal{A} = \{H \in \mathcal{A} \mid X \subset H\}$ .

However, although the homeomorphism type is combinatorially determined, an explicit formula for the torsion in the first homology group is unknown. This problem is also listed in [1] as Problem 24.4.19.

**Problem 2.** *Find a nice formula for the torsion of  $H_1(\partial F; \mathbb{Z})$ .*

There are several partial results to this problem.

**Theorem 3.** (Theorem 1.3 in [2]) *Suppose  $\mathcal{A}$  is a generic arrangement, that is  $|\mathcal{A}_X| = 2$ , for each  $X \in L_2(\mathcal{A})$ . Then,*

$$H_1(\partial F; \mathbb{Z}) \cong \mathbb{Z}^{n(n-1)/2} \oplus \mathbb{Z}_n^{(n-2)(n-3)/2}.$$

**Theorem 4.** (Theorem 1.4 in [3]) *Suppose  $|\mathcal{A}|$  is a power of two and  $|\mathcal{A}_X|$  is two or odd for each  $X \in L_2(\mathcal{A})$ . Then,*

$$\dim_{\mathbb{Z}_2}(\text{Tor}(H_1(\partial F; \mathbb{Z})) \otimes \mathbb{Z}_2) \geq \beta(\mathcal{A})$$

where  $\beta(\mathcal{A})$  is the beta invariant of  $\mathcal{A}$ , which is equal to the Euler characteristic of the projectivized complement.

Note that the beta invariant satisfies  $\beta(\mathcal{A}) = (n-2)(n-3)/2$  for generic arrangements. The author's computation so far leads to the following conjecture.

**Conjecture 5.** *Let  $\mathcal{A}$  be a central hyperplane arrangement in  $\mathbb{C}^3$  with  $n$  hyperplanes and  $\partial F$  be its Milnor fiber boundary.*

(1) Suppose that  $(|A_X| - 2)(\gcd(|A_X|, n) - 1) = 0$  for each  $X \in L_2(\mathcal{A})$ . Then,

$$\text{Tor}(H_1(\partial F; \mathbb{Z})) \cong \mathbb{Z}_n^{\beta(\mathcal{A})}$$

In particular, the torsion part is the direct sum of  $\mathbb{Z}_n$  for any arrangement if  $n$  is prime.

(2) If  $\mathbb{Z}_m$  is contained in the torsion of  $H_1(\partial F; \mathbb{Z})$ , then  $m|n$ .

(3)  $H_1(\partial F; \mathbb{Z})$  is torsion free if and only if  $\mathcal{A}$  is either pencil or near pencil.

There are two major ways to compute the topological invariants of the Milnor fiber boundary.

- Applying Némethi-Szilárd algorithm to obtain the plumbing graph of the Milnor fiber boundary. In the proof of Theorem 3, the author used this technique and studied the incidence matrix and its Smith normal form of the plumbing graph to compute the homology group.
- It is known that the Milnor fiber boundary is a cyclic cover of the boundary manifold of the projectivized line arrangement. Use the covering theoretic approach to study the Milnor fiber boundary. In the proof of Theorem 4, the author used the theory of double coverings and the resonance property of the boundary manifold.

## References

- [1] A. Némethi, A. Szilárd, *Milnor fiber boundary of a non-isolated surface singularity*, Lecture notes in Math. vol. 2037, Springer-Verlag, Berlin Heidelberg (2012).
- [2] S. Sugawara, *First homology groups of the Milnor fiber boundary for generic hyperplane arrangements in  $\mathbb{C}^3$* , Bull. Lond. Math. Soc. 57 (2025), no. 12, 4015-4036.
- [3] S. Sugawara, *Even torsions in the homology group of the Milnor fiber boundary of hyperplane arrangements in  $\mathbb{C}^3$* , preprint, arXiv:2512.01371.

## 7: The minimum period of the equivariant characteristic quasi-polynomials

(suggested by Ryo Uchiumi)

Let  $L \cong \mathbb{Z}^\ell$  be a lattice,  $L_{\mathbb{R}} = L \otimes \mathbb{R}$ , and  $L^\vee = \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$  denote the dual lattice of  $L$ . For fixed elements  $\alpha_1, \dots, \alpha_n \in L^\vee \setminus \{0\}$ , define a hyperplane  $H_i = \{x \in L_{\mathbb{R}} \mid \alpha_i(x) = 0\}$  for  $i \in \{1, \dots, n\}$  and a hyperplane arrangement  $\mathcal{A} = \{H_1, \dots, H_n\}$ . For each  $q \in \mathbb{Z}_{>0}$ , let  $L_q = L/qL$  and

$$M(\mathcal{A}; q) = \{\bar{x} \in L_q \mid \alpha_i(x) \not\equiv 0 \pmod{q}\}.$$

The set  $M(\mathcal{A}; q)$  can be considered as the complement of ‘‘hyperplane arrangement’’ in  $L_q$ . Kamiya–Takemura–Terao proved in [2] that the cardinality of  $M(\mathcal{A}; q)$  is a quasi-polynomial in  $q$ , that is, there exist  $\tilde{n} \in \mathbb{Z}_{>0}$  and polynomials  $f^{(1)}, \dots, f^{(\tilde{n})} \in \mathbb{Z}[t]$  such that

$$\#M(\mathcal{A}; q) = f^{(r)}(q) \quad \text{if } q \equiv r \pmod{\tilde{n}}.$$

The counting function  $\chi_{\mathcal{A}}^{\text{quasi}}(q) = \#M(\mathcal{A}; q)$  is called a *characteristic quasi-polynomial* of  $\mathcal{A}$ . Each polynomial  $f^{(r)}$  is called an *r-th constituent*. It is well known that the constituent  $f^{(r)}$  for  $r$  coprime with  $\tilde{n}$  is equal to the *characteristic polynomial* of  $\mathcal{A}$ . The number  $\tilde{n}$  is called a *period* of  $\chi_{\mathcal{A}}^{\text{quasi}}$ , and the smallest such  $\tilde{n}$  is called the *minimum period* of  $\chi_{\mathcal{A}}^{\text{quasi}}$  and denoted by  $\tilde{n}_{\mathcal{A}}$ .

For fixed basis  $\{\beta_1^\vee, \dots, \beta_\ell^\vee\}$  for  $L^\vee$ , let  $S = (s_{ij})_{ij}$  be an  $\ell \times n$  matrix satisfying

$$\alpha_j = s_{1j}\beta_1^\vee + \dots + s_{\ell j}\beta_\ell^\vee \quad \text{for } j \in \{1, \dots, n\}.$$

For a subset  $J \subseteq \{1, \dots, n\}$ , define  $S_J$  as the submatrix of  $S$  corresponding to  $J$ . Let  $r(J) = \text{rank}(S_J)$ , and  $d_{J,1}, \dots, d_{J,r(J)}$  denote the elementary divisors of  $S_J$ . It is known that

$$\chi_{\mathcal{A}}^{\text{quasi}}(q) = q^\ell + \sum_{\substack{J \subseteq \{1, \dots, n\} \\ J \neq \emptyset}} (-1)^{\#J} \left( \prod_{i=1}^{r(J)} \gcd\{d_{J,i}, q\} \right) q^{\ell-r(J)}.$$

Hence

$$\text{lcm}\{d_{J,i} \mid J \subseteq \{1, \dots, n\}, J \neq \emptyset, i \in \{1, \dots, r(J)\}\}$$

is a period of  $\chi_{\mathcal{A}}^{\text{quasi}}$ . It is called the *lcm-period*. Higashitani–Tran–Yoshinaga proved in [1] that the lcm-period is the minimum period of  $\chi_{\mathcal{A}}^{\text{quasi}}$ .

We consider the characteristic polynomial with the group action. For details, see [3]. Let  $\Gamma$  be a finite group acting linearly on  $L$ . In other words, the group homomorphism  $\rho : \Gamma \rightarrow \text{GL}(L)$  is given. Then the  $\Gamma$ -action  $\rho$  induces  $\Gamma$ -actions  $\rho_q$  on  $L_q$  and  $\rho^\vee$  on  $L^\vee$ , respectively, satisfying

$$\rho_q(\gamma) \circ \pi_q = \pi_q \circ \rho(\gamma), \quad \rho^\vee(\gamma)(\alpha) = \alpha \circ \rho(\gamma)^{-1}$$

for  $\gamma \in \Gamma$  and  $\alpha \in L^\vee$ , where  $\pi_q : L \rightarrow L_q$  is the natural projection.

Suppose that a hyperplane arrangement  $\mathcal{A}$  is invariant under the  $\Gamma$ -action. Since  $M(\mathcal{A}; q)$  is invariant under  $\Gamma$ , we can consider the permutation character of  $\Gamma$  on  $M(\mathcal{A}; q)$ , denoted by  $\chi_{\mathcal{A},q}$ . Then, for each  $\gamma \in \Gamma$ , we have

$$\chi_{\mathcal{A},q}(\gamma) = \#\{x \in M(\mathcal{A}; q) \mid \rho_q(\gamma)(x) = x\}.$$

For the identity element  $1 \in \Gamma$ , we have  $\chi_{\mathcal{A},q}(1) = \chi_{\mathcal{A}}^{\text{quasi}}(q)$ .

**Theorem 1** ([3, Theorem 2.6]).  $\chi_{\mathcal{A},q}$  is a quasi-polynomial in  $q$ .

Note that the constituents of the quasi-polynomial  $\chi_{\mathcal{A},q}$  are polynomials whose coefficients are linear combinations of irreducible characters of  $\Gamma$  with rational coefficients.

For each  $\gamma \in \Gamma$ , let  $R_\gamma$  denote the representation matrix of  $\rho(\gamma)$ . For  $J \subseteq \{1, \dots, n\}$ , define  $S_{\gamma,J} = (R_\gamma - I \ S_J)$  as the matrix partitioned into two matrices  $R_\gamma - I$  and  $S_J$ , where  $I$  is the identity matrix of order  $\ell$ . Then we have the following formula (see [3, (2.2)]):

$$\chi_{\mathcal{A},q}(\gamma) = \sum_{J \subseteq \{1, \dots, n\}} (-1)^{\#J} \left( \prod_{i=1}^{r(\gamma,J)} \gcd\{d_{\gamma,J,i}, q\} \right) q^{\ell - r(\gamma,J)},$$

where  $r(\gamma, J) = \text{rank}(S_{\gamma,J})$  and  $d_{\gamma,J,1}, d_{\gamma,J,2}, \dots, d_{\gamma,J,r(\gamma,J)}$  denote the elementary divisors of  $S_{\gamma,J}$ . Hence

$$\tilde{N} = \text{lcm}\{d_{\gamma,J,i} \mid \gamma \in \Gamma, J \subseteq \{1, \dots, n\}, i \in \{1, \dots, r(\gamma, J)\}\}$$

is a period of  $\chi_{\mathcal{A},q}$ . Let

$$\tilde{n}_\Gamma = \text{lcm}\{d_{\gamma,\emptyset,i} \mid \gamma \in \Gamma \setminus \{1\}, i \in \{1, \dots, r(\gamma, \emptyset)\}\}.$$

Then  $\tilde{n}_\Gamma$  is the minimum period of  $\chi_{\emptyset,q}$  ( $\chi_{\mathcal{A},q}$  for the empty arrangement  $\mathcal{A} = \emptyset$ , see [4, Theorem 2.5]). It is clear that  $\tilde{n}_{\mathcal{A}}$  and  $\tilde{n}_\Gamma$  are divisors of  $\tilde{N}$ .

Let  $\tilde{n}_{\min}$  denote the minimum period of  $\chi_{\mathcal{A},q}$ . Since  $\chi_{\mathcal{A},q}(1) = \chi_{\mathcal{A}}^{\text{quasi}}(q)$ ,  $\tilde{n}_{\mathcal{A}}$  is a divisor of  $\tilde{n}_{\min}$ .

The following are known about  $\tilde{N}$  and  $\tilde{n}_{\min}$ :

- In general,  $\tilde{n}_\Gamma$  is not a divisor of  $\tilde{n}_{\min}$ . In other words,  $\tilde{N}$  is not equal to  $\tilde{n}_{\min}$  in general (see [3, Example 2.9]).
- In general, even if  $\tilde{n}_\Gamma$  is a divisor of  $\tilde{n}_{\min}$ , it does not follow  $\text{lcm}\{\tilde{n}_{\mathcal{A}}, \tilde{n}_\Gamma\} = \tilde{n}_{\min}$  (see [3, Example 2.11]).

**Question 1:** Can we characterize hyperplane arrangements  $\mathcal{A}$  defined on  $L$  invariant under a group  $\Gamma$  such that  $\tilde{n}_\Gamma$  divides a period of  $\chi_{\mathcal{A},q}$ ?

**Question 2:** Can we characterize hyperplane arrangements  $\mathcal{A}$  defined on  $L$  invariant under a group  $\Gamma$  such that  $\chi_{\mathcal{A},q}$  has the minimum period  $\text{lcm}\{\tilde{n}_{\mathcal{A}}, \tilde{n}_\Gamma\}$ ?

**Question 3:** Is it possible to find an explicit formula for the minimum period?

## References

- [1] HIGASHITANI, A., TRAN, T. N., AND YOSHINAGA, M. Period Collapse in Characteristic Quasi-Polynomials of Hyperplane Arrangements. *International Mathematics Research Notices* 2023, 10 (2023), 8934–8963.
- [2] KAMIYA, H., TAKEMURA, A., AND TERAOKA, H. Periodicity of hyperplane arrangements with integral coefficients modulo positive integers. *Journal of Algebraic Combinatorics* 27, 3 (2008), 317–330.
- [3] UCHIUMI, R. The characteristic quasi-polynomials of hyperplane arrangements with actions of finite groups, 2025. arXiv:2508.10236 [math].
- [4] UCHIUMI, R., AND YOSHINAGA, M. The quasi-polynomiality of mod  $q$  permutation representation for a linear finite group action on a lattice. *Journal of Algebra* 689 (2026), 501–518.

## 8: Reconstruction of the oriented matroid from the Varchenko-Gelfand Algebra

(suggested by Masahiko Yoshinaga)

Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be a central real hyperplane arrangement in  $V = \mathbb{R}^\ell$ . The set of connected components (chambers) of the complement  $V \setminus \bigcup_{i=1}^n H_i$  is denoted by  $\text{ch}(\mathcal{A})$ . Varchenko and Gelfand (1987) introduced the ring of functions over  $\text{ch}(\mathcal{A})$ . Namely, for an integral domain  $R$ , let

$$\text{VG}(\mathcal{A}) = \{f : \text{ch}(\mathcal{A}) \rightarrow R\},$$

which is called the *Varchenko-Gelfand (VG) algebra*. There are several special elements in  $\text{VG}(\mathcal{A})$ :

- Let us denote by  $1_C \in \text{VG}(\mathcal{A})$  the characteristic function of a chamber  $C \in \text{ch}(\mathcal{A})$ .  $\{1_C \mid C \in \text{ch}(\mathcal{A})\}$  is the set of primitive idempotent elements.
- For each  $H_i \in \mathcal{A}$ , there are two half-spaces  $H_i^+$  and  $H_i^-$ .  $1_{H_i^\pm} = \sum_{C \subset H_i^\pm} 1_C$  are called the Heaviside functions of  $H_i$ .

Clearly, we have

$$1 = 1_{H_i^+} + 1_{H_i^-} = \sum_{C \in \text{ch}(\mathcal{A})} 1_C.$$

We also have  $\text{VG}(\mathcal{A}) \simeq \mathbb{R}^{\text{ch}(\mathcal{A})} = \prod_{C \in \text{ch}(\mathcal{A})} \mathbb{R} \cdot 1_C$  as  $\mathbb{R}$ -algebras.

It is known that  $\text{VG}(\mathcal{A})$  is generated by Heaviside functions. Let us denote by  $F_k \text{VG}(\mathcal{A})$  the set of  $f \in \text{VG}(\mathcal{A})$  which is expressed as a polynomial of Heaviside functions of degree at most  $k$ . We have the filtration  $0 = F_{-1} \text{VG}(\mathcal{A}) \subset F_0 \text{VG}(\mathcal{A}) = \mathbb{R} \subset F_1 \text{VG}(\mathcal{A}) \subset \dots$  which is called the *Varchenko-Gelfand filtration*. We consider  $\text{VG}(\mathcal{A})$  as the filtered algebra with this filtration. Varchenko and Gelfand proved that the filtered algebra structure of  $\text{VG}(\mathcal{A})$  is related to topology of  $M = \mathbb{C}^\ell \setminus \bigcup_{i=1}^n H_i \otimes \mathbb{C}$ .

- (1)  $F_\ell \text{VG}(\mathcal{A}) = \text{VG}(\mathcal{A})$ .
- (2)  $\text{gr}_k^F \text{VG}(\mathcal{A}) = F_k \text{VG}(\mathcal{A}) / F_{k-1} \text{VG}(\mathcal{A})$  is a free  $\mathbb{R}$  module of rank  $b_k(M)$ . In other words,  $\text{gr}_k^F \text{VG}(\mathcal{A}) \simeq H^k(M, \mathbb{R})$  as  $\mathbb{R}$ -modules.
- (3) When  $\mathbb{R} = \mathbb{F}_2$ , then the graded algebra  $\text{gr}^F \text{VG}^\bullet(\mathcal{A}) = \bigoplus_{i=0}^\ell \text{gr}^F \text{VG}(\mathcal{A})$  is isomorphic to  $H^\bullet(M, \mathbb{F}_2)$  as  $\mathbb{F}_2$ -algebras.

In particular, (3) states that, when the coefficient ring is  $\mathbb{R} = \mathbb{F}_2$ , the graded Varchenko-Gelfand algebra  $\text{gr}^F \text{VG}^\bullet(\mathcal{A})$  has as much information as the Orlik-Solomon algebra.

It is natural to ask the case  $\text{char } \mathbb{R} \neq 2$ . Y. Yagi and myself (arXiv:2509.19905) recently studied such cases. The results are, in short, unlike the case of  $\text{char} = 2$ , filtered and graded VG-algebras would have full information of oriented matroids (up to reorientations). More precisely, suppose  $\text{char } \mathbb{R} \neq 2$  and  $\mathcal{A}$  is generic in codimension 2 (equivalently, distinct three hyperplanes  $H_i, H_j, H_k$  have the intersection  $H_i \cap H_j \cap H_k$  of codimension 3). Then,

- (a) the adjacent graph of chambers can be recovered from the filtered VG-algebra  $VG(\mathcal{A})$ .
- (b) the adjacent graph of chambers can be recovered also from the graded VG-algebra  $\text{gr}^F VG^\bullet(\mathcal{A})$ .

Note that (b) is clearly stronger than (a). However, the direct proof of (a) is much easier than the proof of (b). The proof of (b) requires the presentation of  $\text{gr}^F VG^\bullet(\mathcal{A})$  in terms of generators and relations, however, (a) does not.

The following is a very natural question:

**Question 1:** Can one reconstruct the adjacent graph of chambers from the filtered (resp. graded) algebra  $VG(\mathcal{A})$  (resp.  $\text{gr}^F VG^\bullet(\mathcal{A})$ ) for any arrangement?

The difficulty of the question is also related to the structure of the automorphism group of the filtered algebra  $VG(\mathcal{A})$ . We proved (in arXiv:2509.19905) that under the assumption “generic in codimension 2”, the automorphism group  $\text{Aut}(VG(\mathcal{A}))$  of the filtered algebra  $VG(\mathcal{A})$  is isomorphic to the automorphism group of the adjacent graph. However,  $\text{Aut}(VG(\mathcal{A}))$  can become bigger than the automorphism group of the adjacent graph.

**Question 2:** When  $\text{Aut}(VG(\mathcal{A}))$  is isomorphic to the group of the adjacent graph?

The following are also open questions.

**Question 3:** Suppose  $\text{char } R \neq 2$ . Can one recover the filtered VG-algebra  $VG(\mathcal{A})$  from the graded VG-algebra  $\text{gr}^F VG^\bullet(\mathcal{A})$ ?

If  $\mathcal{A}$  is generic in codimension 2, then the answer to the above question is “yes”. Indeed, one can recover the adjacency graph of chambers from  $\text{gr}^F VG^\bullet(\mathcal{A})$  by (b), then construct  $VG(\mathcal{A})$  from the oriented matroid. It is of course desired to recover  $VG(\mathcal{A})$  more directly from  $\text{gr}^F VG^\bullet(\mathcal{A})$ .

**Question 4:** Suppose  $R = \mathbb{F}_2$ . Can one recover the adjacency graph from  $VG(\mathcal{A})$ ?

The answer to this question is likely No. However, we do not know explicit examples.

## 9: Enumerating chambers in $m = 1$ Grasstopes

(suggested by Francesca Zaffalon)

Given  $n, k, m$  positive integers with  $k + m \leq n$  and  $Z$  an  $n \times (k + m)$  matrix, consider the (rational) map

$$\tilde{Z} : \text{Gr}_{\geq 0}(k, n) \dashrightarrow \text{Gr}(k, k + m), \quad C \mapsto C \cdot Z.$$

The image  $\tilde{Z}(\text{Gr}_{\geq 0}(k, n))$  is called a *Grasstope* and is denoted by  $\mathcal{G}_{n,k,m}(Z)$ . If, in addition, all  $(k + m) \times (k + m)$  minors of  $Z$  are positive, then the map  $\tilde{Z}$  is well-defined and the image  $\mathcal{A}_{n,k,m}$  is called *amplituhedron*. The latter was introduced in physics [1], as a geometric framework to encode scattering amplitudes.

To describe Grasstopes via inequalities, we will use *twistor coordinates*. Given  $U \in \text{Gr}(k, k + m)$  and  $I \in \binom{[n]}{m}$ , the  $I$ -th twistor coordinates of  $U$  is given by  $\langle Z, I \rangle(U) = \det \left( \frac{U}{Z_I} \right)$ , where  $Z_I$  is the submatrix of  $Z$  given by rows labeled by  $I$ .

When  $m = 1$ , amplituhedra and Grasstopes live in a projective space  $\text{Gr}(k, k + 1) \cong \mathbb{P}^k$ . The  $m = 1$  amplituhedron  $\mathcal{A}_{n,k,1}(Z)$  is equal to the the complex of bounded regions in the cyclic hyperplane arrangement defined by  $Z$  [2]. More generally, when the map  $\tilde{Z}$  is well-defined on  $\text{Gr}_{\geq 0}(k, n)$ , the  $m = 1$  Grasstopes  $\mathcal{G}_{n,k,1}(Z)$  consist of points  $x \in \mathbb{P}^k$  such that

$$\overline{\text{var}}(\langle Z_1 \rangle(x^\perp), \dots, \langle Z_n \rangle(x^\perp)) \geq k,$$

where  $\overline{\text{var}}(u)$  denotes the maximal number of sign changes in  $u'$ , where  $u'_i = u_i$  whenever  $u_i \neq 0$ . For example,  $\overline{\text{var}}((1, 0, 1, 2, -1)) = 3$  and  $\overline{\text{var}}((1, 2, 0, -1, -5)) = 1$  [3].

In [3], the study of the number of regions in a given Grasstope was initiated. Note that this number is bounded between

$$m(n, k) = \frac{1}{2} \left( \binom{n-1}{k} + \sum_{i=0}^k \binom{n}{i} \right) - \sum_{i=0}^{k-1} \binom{n-1}{i},$$

$$M(n, k) = \min \left( \frac{1}{2} \left( \binom{n-1}{k} + \sum_{i=0}^k \binom{n}{i} \right), 2^{n-1} - \sum_{i=0}^{k-1} \binom{n-1}{i} \right).$$

The  $m = 1$  amplituhedron always achieves the minimum number of regions  $m(n, k)$ .

**Question 1:** Is it always possible to find  $Z$ s such that the corresponding Grasstope has the maximal number of regions? Is it always possible to find such  $Z$ s arising as a reorientation and reordering of a totally positive matrix?

One can use the description of  $m = 1$  Grasstopes given above to generalize the definition to any oriented matroid. Given  $\mathcal{M}$  an oriented matroid on  $[n]$  of rank  $k + 1$ , define the corresponding Grasstope  $\mathcal{G}_{n,k,1}(\mathcal{M})$  to be the set of covectors  $x$  such that  $\overline{\text{var}}(x) \geq k$ .

**Question 2:** What is the minimum and maximum number of regions in  $\mathcal{G}_{n,k,1}(\mathcal{M}')$  for  $\mathcal{M}'$  in the isomorphism class of  $\mathcal{M}$ .

## References

- [1] ARKANI-HAMED, N., AND TRNKA, J. The amplituhedron. *Journal of High Energy Physics* 2014, 10 (2014), 1–33.
- [2] KARP, S. N., AND WILLIAMS, L. K. The  $m = 1$  amplituhedron and cyclic hyperplane arrangements. *International Mathematics Research Notices* 2019, 5 (2019), 1401–1462.
- [3] MANDELSHTAM, Y., PAVLOV, D., AND PRATT, E. Combinatorics of  $m = 1$  Grasstopes. *Combinatorial Theory* 5, 2 (2025).